Models for the Near-Surface Oceanic Vorticity and Divergence

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Abstract - Model-based scatterometer wind retrieval algorithms are based on parametric models for the near-surface wind field. Crucial to these models are the representation of the wind vorticity and divergence fields. As part of an effort to improve the modeling accuracy of these fields, the spectra of the vorticity and divergence fields has been computed using ERS-1 scatterometer winds. Over scales of 100 km to 1000 km the vorticity and divergence fields exhibit a power-law dependence on wavenumber k of approximately $k^{-2.6}$ and $k^{-1.5}$, respectively. This suggests that low-order numerical models can be used to model these fields with the level of accuracy required for model-based wind retrieval. We apply the Karhunen-Loeve (KL) transform to develop data-derived statistical models for the vorticity and divergence fields and compare the resulting wind field model to a previous model based on a polynomial expansion. The KL-based model provides some improvement in the model accuracy.

INTRODUCTION

Spaceborne scatterometers provide global measurements of near-surface oceanic vector winds. The scatterometer measures the normalized radar backscatter (σ^o) of the ocean surface. From multiple measurements of σ^o vector winds can be determined using a geophysical model function relating wind and backscatter and a wind retrieval algorithm.

Traditionally, the retrieval algorithms use only the σ^o measurements associated with a given sample point to estimate the wind at that point. This pointwise method produces a set of possible wind vectors at each sample point and so requires an additional "dealiasing" step to select a unique estimate at each point.

Recently, a parametric model-based approach to estimate the vector wind field has been developed. By using a model for the wind field, this approach can avoid point-wise dealiasing. The model is based on simplified equations of motion of the near-surface wind. Wind vector components are written in terms of the pressure field boundary conditions and the vorticity and divergence fields which are approximated by low-order polynomials. The coefficients and boundary conditions are estimated using maximum-likelihood (ML) techniques from the σ^o measurements. In an effort to improve models of the wind vorticity and divergence fields, we have examined the spatial variability of the vorticity and divergence fields using one dimensional spectra.

In this paper we report recent empirical results describing the one-dimensional wavenumber spectra of the vorticity and divergence fields. We also consider data-derived, statistically "optimal" models for these fields as an alternative to the polynomials previously used.

VORTICITY AND DIVERGENCE SPECTRA

The ERS-1 scatterometer provides 50 km resolution winds with a 25 km sample spacing over a 500 km wide swath. While the wind measurements are given on a rectilinear grid, the grid is tilted with respect to North, complicating the computation of the vorticity and divergence from the wind vectors. Winds are retrieved from ERS-1 σ^o using both pointwise and model-based techniques. The C-band model function of Freilich and Dunbar (1993) is used. The ambiguity selected for the pointwise result is determined from the JPL value-added product (Freilich and Dunbar, 1993).

Vorticity and divergence fields are produced as auxiliary products of model-based estimation. However, for this study the vorticity and divergence are computed from the estimated wind vector field using a first difference approximation to the derivative. This permits both pointwise and model-based winds to be inputs. Similar results were obtained for both. Using a difference approximation necessitates taking the differences in the along-track and cross-track directions. A rotation is applied to the u and v component vectors prior to computing the derivative to avoid coordinate system inconsistencies.

The vorticity and divergence fields were extracted from ERS-1 passes over the Pacific Ocean in the five latitude bands shown in Table 1. Data passes with gaps or land were discarded. A total of 40 orbit passes were used in this preliminary result.

TABLE 1. *Study Regions*

Region	Latitude (° North)	Longitude (° East)
1	-45° to -25°	160° to 280°
2	-25° to -5°	160° to 280°
3	5° to 25°	140° to 250°
4	25° to 45°	150° to 230°
5	-5° to 5°	150° to 280°

For this paper we consider only one-dimensional spectra in the along-track direction. The spectra were separately computed for each study region by first computing the along-track autocorrelation function for each cross-track bin and pass. The autocorrelation estimates are then averaged in the cross-track direction and finally over the passes to provides an estimate of the "expected" autocorrelation function. The wavenumber spectra are computed as the FFT of the averaged autocorrelation estimates. Figures 1 and 2 illustrate the observed wavenumber spectra of the vorticity and divergence for each study region, respectively. The vertical scale is arbitrary.

The most important observation seen in these figures is that over spatial scales of about 100 to 1000 km the vorticity and divergence fields each exhibit a power-law dependence on wavenumber k of the form k^p. The value of p is determined over a wavenumber range corresponding to 100 to 1000 km using linear regression in log-log space. For the vorticity field, $-2.66 < p < -2.5$, depending on the region. The power-law fit for the divergence field is somewhat dependent on the spatial range of the regression; however, for 100 to 1000 km, $-1.64 < p < -1.4$. These power-law coefficients for the vorticity and divergence fields may be compared to the $p = 2$ power-law dependence of the wind component fields over the same scale range (Freilich and Chelton, 1986).

The low-pass nature of the vorticity and divergence fields suggests that low-order coefficients of a series expansion model will dominate the other terms in the series. This can be exploited to develop low-order models which provide suitable accuracy for model-based wind retrieval.

VORTICITY AND DIVERGENCE MODELLING

The vorticity and divergence fields of the near surface oceanic vector wind $\mathbf{U} = [u, v]^T$ are defined by,

$$\begin{aligned} \zeta &= \mathbf{k} \cdot \nabla \times \mathbf{U} \quad \text{Vorticity} \\ \delta &= \nabla \cdot \mathbf{U}. \quad \text{Divergence} \end{aligned} \tag{1}$$

Using the vector Helmholtz theorem, the wind $\mathbf{U}$ may be decomposed into a nondivergent and a curl-free vector field,

$$\mathbf{U} = \underbrace{\mathbf{k} \times \nabla\psi}_{\text{nondivergent}} + \underbrace{\nabla\chi.}_{\text{curl-free}} \tag{2}$$

Taking the divergence and curl of Eq. (2) two Poisson equations result,

$$\nabla^2\psi = \zeta \tag{3}$$

$$\nabla^2\chi = \delta. \tag{4}$$

Assuming that $\chi = 0$ on the boundary and that over the region of interest ψ corresponds to the geostrophic pressure field, the wind field components can be expressed as

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$$u = -\frac{\partial\psi}{\partial y} + \frac{\partial\chi}{\partial x} \quad (5)$$

$$v = \frac{\partial\psi}{\partial x} + \frac{\partial\chi}{\partial y}. \quad (6)$$

To develop a simple wind field model Long (1993) approximated the vorticity and divergence fields as low-order bivariate polynomial fields of the form,

$$\zeta(x,y) = \sum_{m=0}^{M_v} \sum_{\substack{n=0 \\ m+n\leq M_v}}^{M_v} v_{m,n} x^m y^n \quad (7)$$

$$\delta(x,y) = \sum_{m=0}^{M_d} \sum_{\substack{n=0 \\ m+n\leq M_d}}^{M_d} d_{m,n} x^m y^n \quad (8)$$

where M_v and M_d are the vorticity and divergence model orders. In model-based wind retrieval Eqs. (3)-(8) are discretized and solved over a rectangular region, typically 250 km × 250 km. The wind field is a function of the model parameters: the pressure field along the region boundary and the the coefficients $v_{m,n}$ and $d_{m,n}$. These model parameters are estimated directly from the scatterometer σ^o measurements. The resulting wind field model accuracy is dependent on the model order for the vorticity and divergence fields.

The model for the vorticity and divergence need not be exact since noise in the σ^o measurements is a primary source of error in the retrieved winds and a tradeoff between model accuracy (order) and the measurement noise level can be made. A low-order model is desired to minimize computation. Model orders of 2 or 3 have been found to be adequate for wind retrieval from Seasat scatterometer data (Long, 1993).

We may view the operation of approximating a field by a series of polynomials as reconstructing a field from a projection on to a subspace spanned by the polynomials in the series. Finding the series coefficients is analogous to finding the subspace coordinates. Viewed this way, the question of an optimal basis for representing the vorticity and divergence fields arises. One criteria of optimality is the mean-squared reconstruction error. Of all series expansions, the Karhunen-Loeve (KL) transform minimizes the average mean-squared error. Since many readers are not familiar with this transform, we provide a brief overview of the KL transform.

KARHUNEN-LOEVE EXPANSION

The *Karhunen-Loeve* (KL) *transform* of a wide-sense stationary (WSS) finite-support, discrete random field x is given by (Jain, 1989):

$$\mathcal{Y} = \Psi^H \mathcal{X} \quad (9)$$

where $\mathcal{X} = \mathtt{vec}(x)$ and superscript H represents Hermitian transpose (complex conjugate transpose). The KL transform is a projection onto the KL basis set. The KL estimate of a field is reconstructed from the projection by,

$$\hat{\mathcal{X}} = \Psi^H \mathcal{Y}. \quad (10)$$

where the KL transform matrix Ψ is the solution of the eigenequation

$$R_x \Psi = \Psi \Lambda \quad (11)$$

where $R_x = E(\mathcal{X}\mathcal{X}^H)$ is the autocorrelation matrix of the random field x. Since x is WSS, R_x is also doubly block Toeplitz (Jain, 1989).

The KL transform has several useful properties that are exploited in our results. These are: (Jain, 1989)

1. The columns of Ψ are orthonormal (*i.e.*, Ψ is unitary with $\Psi^H\Psi = \Psi\Psi^H = I$ or $\Psi^{-1} = \Psi^H$) and the columns of Ψ form a complete basis. This property makes possible computing the KL series coefficients by one matrix-vector multiplication. In contrast, the polynomials used by Long (1993) are non-orthogonal.
2. The KL transform represents the observed data $\mathcal{X}$ in a space where it is spatially uncorrelated. The columns of Ψ are a complete orthonormal basis for this special space.
3. On average, over all possible series approximations, the KL transform has the minimum mean-square (MS) error.

KL-BASED MODELS

Since the KL expansion has the smallest MS error on average among all series expansions, replacing the polynomial series in the wind field model by the KL series should give more accurate wind field models and therefore more accurate model-based wind estimates.

Recall that the KL basis (columns of Ψ) depend upon having knowledge of the true autocorrelation matrix R. In practice, however, this matrix is unavailable and is replaced by an estimate of the autocorrelation matrix.

In computing R_x, we only need to look at only small subregions, e.g., 250 km × 250 km (10×10), for which the model will be applied in model-based estimation. Using the same passes as in the previous study, 10×10 subregions are extracted and the two-dimensional sample autocorrelation function computed. The subregions overlap in both the along-track and cross-track dimensions. The sample autocorrelation matrix is constructed from the sample autocorrelation function using a block Toeplitz extension. This sample autocorrelation matrix is then used to derive the KL basis set using Eq. (11).

Using the complete KL basis set results in exact field reconstruction. However, since an exact model is not required in model-based estimation we choose a subset of the basis set to reconstruct the field. The subset is a truncated series of the KL basis vectors sorted by their corresponding eigenvalues.

To compare the KL series model of the vorticity and divergence fields to the polynomial series, models of various order and type were each incorporated into a wind field model. Each wind field model is fit directly to a series of test wind fields using least-squares and the root-mean-squared reconstruction error computed.

The results are summarized in Figures 3 and 4. To generate Fig. 3, the order of the vorticity model was varied while the order of the divergence model was held fixed. Similarly, to generate Fig. 4 the order of the divergence model was varied while the order of the divergence model was held fixed. In both models we see a rapid decrease in modeling error with model order. This is the result of the low-pass nature of the vorticity and divergence fields. We note that although the KL model has uniformly lower error in the range shown, the difference is small. This result is initially surprising given the theoretic optimality of the KL expansion. However, the polynomials bear a remarkable resemblance to the KL basis functions. Even so, the KL basis models do result in improved accuracy in model-based wind retrieval.

SUMMARY

In this paper, we have presented one-dimensional spectral plots of the mesoscale wind vorticity and divergence. These exhibit a power-law dependence on wavenumber similar to $k^{-2.6}$ and $k^{-1.5}$, respectively, over spatial scales from 100 km to 1000 km. We have also used the KL transform to develop an "optimal" statistical model for the vorticity and divergence fields based on the eigenstructure of estimated autocorrelation functions. This model is a small improvement over a low-order polynomial model in terms of mean-squared reconstruction error but does result in more accurate wind field estimates.

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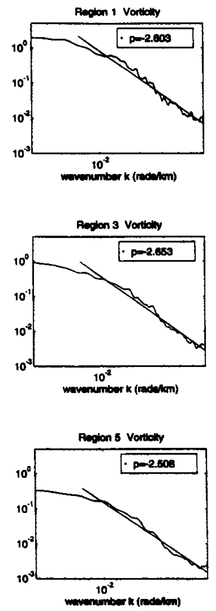

Figure 1.
Observed one-dimensional spectra of the wind vorticity field for each study region. The vertical scale is arbitrary.

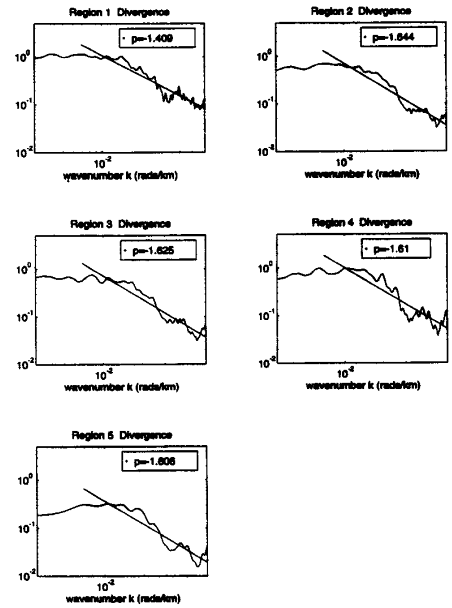

Figure 2.
Observed one-dimensional spectra of the wind divergence field for each study region. The vertical scale is arbitrary.

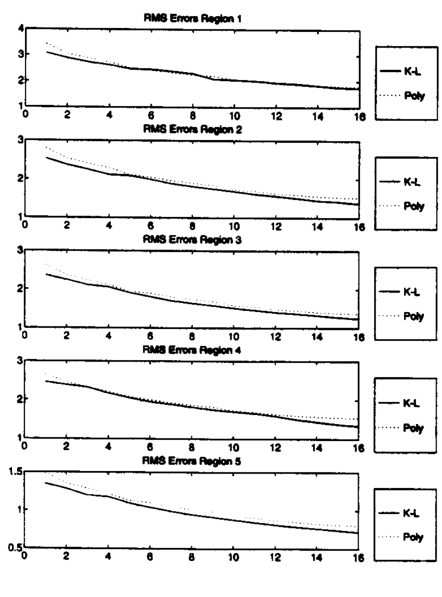

Figure 3.
The rms wind field fit error (in m/s) as a function of the number of coefficients in the KL and polynomial vorticity models for each study region. The divergence field model is a 16 coefficient polynomial.

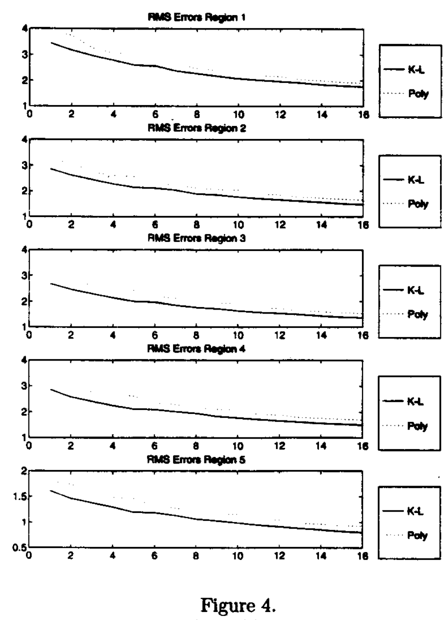

Figure 4.
The rms wind field fit error (in m/s) as a function of the number of coefficients in the KL and polynomial divergence models for each study region. The vorticity field model is a 16 coefficient polynomial.