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AN EXACT NUMERICAL ALGORITHM FOR COMPUTING THE UNWRAPPED PHASE OF A FINITE-LENGTH SEQUENCE

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ABSTRACT

A direct relationship between a one-dimensional time series and its unwrapped phase was first shown by McGowan and Kuc [1]. They proposed an algorithm for computing the unwrapped phase by counting the number of sign changes in a Sturm sequence generated from the real and imaginary parts of the DFT. Their algorithm is limited to relatively short sequences by numerical accuracy. In this paper an extension of their algorithm is proposed which, by using all integer arithmetic, permits exact computation of the number of multiples of π which must be added to the principle value of the phase to uniquely give the unwrapped phase of a one-dimensional rational-valued finite-length sequence of arbitrary length. This extended algorithm should be of interest when highly accurate phase unwrapping is required.

1. INTRODUCTION

McGowan and Kuc [1] first showed that the number of multiples of π which must be added to the principle value of the phase to obtain the continuous unwrapped phase can be uniquely determined by counting the number of sign changes in a Sturm sequence generated from a finite-length real sequence. Unfortunately, the numerical accuracy required for computation of the Sturm coefficients and evaluation of the Sturm sequence precludes the application of their algorithm beyond relatively short time sequences. In this paper, an extension to this approach is described which uses all integer arithmetic to permit exact numerical computation of the multiples of π (denoted $L(\omega)$) which must be added to the principle value of the phase to obtain the unwrapped phase when the one-dimensional finite-length real sequence is rational-valued. The sequence may be of arbitrary length.

This paper is organized as follows. First, the Sturm sequence method of computing $L(\omega)$ first proposed by McGowan and Kuc is summarized. This is followed by a brief discussion of the limitations in the numerical accuracy of the approach. Next, extensions to this method are provided which permit the use of all-integer arithmetic to exactly compute the coefficients of the Sturm polynomial sequence and to insure the accuracy of the computation of $L(\omega)$. A brief discussion of the tradeoffs in memory and computation required versus accuracy is presented. The appendix presents an important theorem in the application of Sturm sequences to phase unwrapping. A companion paper extends this one-dimensional result to multidimensional sequences and demonstrates the uniqueness of the phase of a multidimensional sequence with a rational **Z** transform [2].

2. STURM SEQUENCE PHASE UNWRAPPING

The DFT, $X(\omega)$, of the real-valued, finite-length time sequence $\{x(n),\ n = 0, ..., N-1\}$ is,

$$X(\omega) = \sum_{n=0}^{N-1} x(n) e^{-jn\omega}.$$

Assuming that $|X(\omega)| \neq 0$, the phase of $X(\omega)$ relative to the phase at $\omega = 0$, is,

$$\arg[X(\omega)] - \arg[X(0)] = \arctan\left\{\frac{Im[X(\omega)]}{Re[X(\omega)]}\right\} - L(\omega)\pi.$$

The integer-valued function $L(\omega)$ indicates the number of multiples of π which must be added to the principle value of the phase of $X(\omega)$ to produce a continuous phase function, i.e., the unwrapped phase. Figure 1 shows a plot of the of $arctanx + \pi l$ for several values of l [1]. As ω increases through a zero of $Re[X(\omega)]$, the ratio, $y(\omega) = Im[X(\omega)]/Re[X(\omega)]$ passes through a pole at either $-\infty$ or $+\infty$. ($X(\omega)$ is non-zero on the unit circle so $Im[X(\omega)]$ and $Re[X(\omega)]$ are not both simultaneously zero.) At this pole, the integer-valued $L(\omega)$ must increase or decrease in order to maintain a continuous phase function (which is the unwrapped phase). $L(\omega)$ will increase if $Im[X(\omega)]/Re[X(\omega)]$ goes from positive to negative as ω goes through a root of $Re[X(\omega)]$ and will decrease if $Im[X(\omega)]/Re[X(\omega)]$ goes from negative to positive through a root of $Re[X(\omega)]$ (see Figure 1). In regions where $Im[X(\omega)]/Re[X(\omega)]$ does not change sign, $L(\omega)$ does not change value. Thus, computing $L(\omega)$ is reduced to computing the sign changes in $Im[X(\omega)]/Re[X(\omega)]$ at the roots of $Re[X(\omega)]$ [1]. This may be accomplished by using a Sturm polynomial sequence computed from the real and imaginary parts of $X(\omega)$.

The real and imaginary parts of $X(\omega)$ can be conviently expressed in terms of Chebychev polynomials of the second kind using the fundamental Chebychev polynomial equations [3],

$$T_n(\omega) = \cos n\omega, \quad U_n(\omega) = \frac{\sin(n+1)\omega}{\sin\omega}, \tag{1}$$

$$T_n(\omega) = \frac{1}{2}[U_n(\omega) - U_{n-2}(\omega)] \ \ \forall n \geq 2,$$

$$T_1(\omega) = \frac{1}{2}U_1(\omega), \quad T_0(\omega) = U_0(\omega) = 1,$$

as,

$$X(\omega) = e^{j(1-N)\omega}\left[\sum_{n=0}^{N-1} p_0(n)U_n(\omega) + j\sin\omega \sum_{n=0}^{N-2} p_1(n)U_n(\omega)\right] \tag{2}$$

with

$$p_0(n) = \begin{cases} x(N) - x(N-2)/2 & n = 0 \\ [x(N-n) - x(N-n-2)]/2 & n \leq N-2 \\ x(N-n)/2 & N-1 \leq n \leq N \end{cases} \quad (3)$$

$$p_1(n) = x(N-n-1) \quad 0 \leq n \leq N-1. \quad (4)$$

Note that $T_n(\omega)$ and $U_n(\omega)$ are polynomials in $\arccos\omega$. The $e^{j(1-N)\omega}$ term in (2) adds a linear phase term to the phase of the term in brackets and is included for convenience.

The $\sin\omega$ term in equation (2) does not affect the sign or the roots of $Im[X(\omega)]$ on the interval $[0,\pi]$ nor the location of the roots of $Re[X(\omega)]$. Thus, it does not affect $L(\omega)$ and need not be considered in the computation of $L(\omega)$. A Sturm sequence of Chebychev polynomials $\{P_0(\omega), P_1(\omega), \ldots, P_m(\omega),\ m = N-1\}$ can be generated from the terms of (2) which permits computation of $L(\omega)$. Define the first two polynomials of the Sturm sequence,

$$P_0(\omega) = \sum_{n=0}^{N-1} p_0(n)U_n(\omega)$$

$$P_1(\omega) = \sum_{n=0}^{N-2} p_1(n)U_n(\omega).$$

where the remaining polynomials of the Sturm sequence are generated from the "negative remainder" relationship:

$$P_{k-1}(\omega) = Q_k(\omega)P_k(\omega) - P_{k+1}(\omega)$$

such that the order of the $(k-1)$th element of the Sturm sequence is less than the k th element. Following [1] $Q_k(\omega)$ is defined as,

$$Q_k(\omega) = q_kU_1(\omega) + r_kU_0(\omega).$$

Then, using the recursive relationship between Chebychev polynomials of the second kind,

$$U_{n+1}(\omega) = U_n(\omega)U_1(\omega) - U_{n-1}(\omega) \quad n \geq 1$$

$$U_0(\omega) = 1$$

the following is obtained,

$$q_k = \frac{p_{k-1}(N-k)}{p_k(N-k-1)} \quad (4)$$

$$r_k = \frac{p_{k-1}(N-k-1) - q_kp_k(N-k-2)}{p_k(N-k-1)} \quad (5)$$

$$P_{k+1}(\omega) = \sum_{n=0}^{N-k-2} p_{k+1}(n)U_n(\omega) \quad (6)$$

$$p_{k+1}(n) = \begin{cases} p_{k-1}(0) - r_kp_k(0) - q_kp_k(1), & n = N-k, \\ p_{k-1}(n) - r_kp_k(n) - \\ q_k[p_k(n-1) + p_k(n+1)], & 1 \leq n \leq N-k-2. \end{cases} \quad (7)$$

with $P_{k+1}(\omega)$ defined,

$$P_{k+1}(\omega) = \sum_{n=0}^{N-k-1} p_{k+1}(n)U_n(\omega),$$

The polynomial division algorithm indicated in equations (4) through (7) is repeated until $P_m(\omega)$, $m = N-1$, contains only constant $U_0(\omega)$ terms.

By the theorem presented in the appendix, the difference between the number of variations of sign in the Sturm sequence evaluated at ω_1 and the Sturm sequence polynomials evaluated at ω_2 ($0 \leq \omega_1 < \omega_2 \leq \pi$) gives the number of positive to negative changes in the sign of $P_1(\omega)/P_0(\omega)$ through the zeros of $P_0(\omega)$ minus the number of positive to negative changes in sign on $[\omega_1, \omega_2]$, i.e. $L(\omega_2) - L(\omega_1)$. Note that $L(\omega)$ is uniquely specified by $P_0(\omega)$ and $P_1(\omega)$ and thus by $x(n)$. While ω_2 and ω_1 can be arbitrarily chosen on $[0,\pi]$, they are typically chosen at equally spaced intervals corresponding to FFT spacing.

This approach to computing the unwrapped phase clearly indicates that the unwrapped phase is unique in the sense that once a value for the phase at $\omega = 0$ is determined, all other values follow. In a later section it will be shown that if $x(n)$ is rational-valued, $L(\omega)$ can be exactly computed using integer arithmetic.

3. NUMERICAL CONSIDERATIONS

Direct application of the algorithm described above can result in numerical problems for long sequences. These problems exist because more digits of significance are required to represent the coefficients of the Sturm sequence polynomials than are available in ordinary or double precision floating point representations used in such high level languages such as FORTRAN or C. Inaccuracies in the computation of the Sturm sequence polynomial coefficients and evaluation at a particular ω due to the loss of least significant digits during floating point multiplication and division can result in the value of the evaluated polynomial having an incorrect sign. When the number of sign changes in the Sturm sequence is counted, an incorrect value for $L(\omega)$ will result.

The sensitivity of this algorithm to numerical accuracy can be emperically observed as the sequence length is extended. For time sequence lengths longer than 20-40 points using ordinary floating point representations, the computed unwrapped phase estimate is very often incorrect.

The next section presents a technique to eliminate the underflow problems associated with the use of floating point computation by using all-integer arithmetic.

4. ALGORITHM EXTENSION

The extension described in this section assumes that the time sequence takes on only rational values. This is considered to be a relatively mild restriction since signals are typically digitized to integer values. Note that a sequence of rational values can always be expressed as a sequence of integers with a common divisor. The divisor does not affect the phase function and can be ignored. Thus, without loss of generality, the time sequence can be further restricted to strictly integer values.

For integer-valued time sequences, the Chebychev polnomial coefficients in (3) are integers with a multiplicative constant of $\frac{1}{2}$. However, since only the signs of the Sturm sequence polynomials are of interest, any of the Sturm sequence polynomials can be multiplied by positive constants without affecting the result. Thus, the multiplicative factor of $\frac{1}{2}$ can be discarded so that $p_0(n)$ can be redefined as,

$$p_0(n) = \begin{cases} 2x(N) - x(N-2), & n = 0, \\ x(N-n) - x(N-n-2), & n \leq N-2, \\ x(N-n), & N-1 \leq n \leq N. \end{cases}$$

By the same reasoning, the polynomial division algorithm in equations (4) through (7) can be modified to eliminate the divisions in equations (4) and (5) by scaling them by the positive constant $p_k^2(N-k-1)$. Equations (4) through (7) then become,

$$
\begin{aligned}
q_k &= p_{k-1}(N-k)p_k(N-k-1) \\
r_k &= p_{k-1}(N-k-1)p_k(N-k-1) \\
&\quad -p_{k-1}(N-k)p_k(N-k-2) \\
p_{k-1}(n) &= \begin{cases} -p_{k-1}(0)p_k^2(N-k-1) & \\ \quad +r_kp_k(0)+q_kp_k(1), & n=N-k, \\ -p_{k-1}(n)p_k^2(N-k-1) & \\ \quad +r_kp_k(n)+q_k[p_k(n-1) & \\ \quad +p_k(n+1)], & 1\le n\le N-k-2. \end{cases}
\end{aligned}
$$

The resulting modified Sturm sequence consists of polynomials with strictly integer coefficients. If integer overflow is avoided, the integer coefficients can be computed exactly. Since the polynomials can be arbitrarily scaled without affecting the results so long as overflow or underflow errors are avoided, the number of bits required to represent the coefficients can be reduced, at the expense of additional CPU time, by removing common factors of the coefficients of $P_k(\omega)$ until they are relatively prime. This does not affect the sign changes in the resulting Sturm sequence and, hence, $L(\omega)$.

Having shown that the coefficients of the Sturm sequence polynomials can be exactly computed using integer arithmetic, I now demonstrate an approach to evaluating the Sturm polynomial sequence with sufficient accuracy using-all-integer arithmetic to guarantee that the elements of the evaluated Sturm sequence have the correct sign.

The inherently large, $(-\infty,\infty)$, dynamic range of $U_n(\omega)$ complicates a numerical algorithm. However, note that the denominator of $U_n(\omega)$, sin ω, is independent of n (see (1)) and is positive for $0\le\omega\le\pi$. Thus,

$$
V_n(\omega) \triangleq \begin{cases} n+1, & \omega=0, \\ \sin[(n+1)\omega], & \omega\neq 0. \end{cases} \tag{18}
$$

can be used in place of $U_n(\omega)$ without effecting the number of sign changes in the Sturm sequence. The smaller range of $V_n(\omega)$ reduces the propagation of numerical errors if the decision is made to truncate the polynomial coefficients to reduce storage requirements (discussed below). $V_n(\omega)$ can be obtained to the desired significance by computation of the sine to the desired accuracy or from a sine table. This solves the problem of obtaining a sufficient number of digits of significance for $U_n(\omega)$. $V_n(\omega)$ also allows us to exploit the circular symmetry of the sine function when evaluating the Sturm sequence polynomials to reduce the computation required as well as reduce the number of distinct values of $V_n(\omega)$ which must be computed and stored.

Evaluation of the Sturm sequence polynomials can be done with all-integer arithmetic by using a D-digit truncated integer representation of $V_n(\omega)$, i.e., by defining,

$$
Vi_n(\omega) = \text{Nearest Integer}\{V_n(\omega)10^D\}.
$$

The fact that $|Vi_n(\omega)-V_n(\omega)10^D|\le\frac{1}{2}$ can be exploited to check the accuracy of the polynomial evaluation to insure that a sufficient number of digits in $V_n(\omega)$ have been retained to permit accurate determination of the sign of the evaluated polynomial. Note that when w is a multiple of $\frac{\pi}{2}$, $V_n(\omega)$ is an integer and the Sturm sequence polynomials can be exactly evaluated using integer arithmetic.

For $w\neq 0$ define,

$$
\begin{aligned}
P_k(\omega) &= \sum_{n=0}^{N-k} p_k(n)V_n(\omega)10^D \\
P_k^i(\omega) &= \sum_{n=0}^{N-k} p_k(n)Vi_n(\omega) \\
A_k &= \sum_{n=0}^{N-k} a_k(n) \\
a_k(n) &= \begin{cases} 0, & |V_n(\omega)|=1 \text{ or } 0, \\ |p_k(n)|, & \text{otherwise.} \end{cases}
\end{aligned}
$$

$P_k(\omega)$ represents the ideal value for the evaluated Sturm polynomial element while $P_k^i(\omega)$ is an integer-valued approximation. Note that terms for which $|V_n(\omega)|=1$ or 0 can be computed exactly. A_k provides error bounds for $P_k(\omega)$,

$$
|P_k^i(\omega)|-A_k\le|P_k(\omega)|\le|P_k^i(\omega)|+A_k.
$$

If $|P_k^i(\omega)|>A_k$ then $P_k^i(\omega)$ will have the correct sign. If, however, $|P_k^i(\omega)|\le A_k$ then the correctness of the sign of $P_k^i(\omega)$ can not be guaranteed.

When evaluating the Sturm sequence, A_k can be computed and checked against $|P_k^i(\omega)|$. When this check fails, i.e., $A_k\ge|P_k^i(\omega)|$ for any k, D must be increased to guarantee that $P_k^i(\omega)$ has the correct sign. The minimum D to guarantee the correct sign of $P_k^i(\omega)$ depends on the value of ω as well as sequence.

The number of sign changes in the modified Sturm sequence, $\{P_k^i(\omega),\ k=1,N-1\}$, at $\omega>0$ and $\omega=0$ is used to compute $L(\omega)$. The principle value of the phase at ω can be computed using the first two terms of the modified Sturm sequence evaluated at ω and the additive linear phase term in (2). The unwrapped phase is,

$$
\begin{aligned}
&\arg[X(\omega)]-\arg[X(0)] = \\
&\quad -\arctan\Big\{2\frac{P_1^i(\omega)}{P_0^i(\omega)}\sin\omega\Big\}+L(\omega)\pi+(N-1)\omega.
\end{aligned}
$$

5. ACCURACY VERSUS COMPUTATION

For extremely long sequences, the number of bits required to exactly represent the integer coefficients may become very large. Since ultimately only the sign of the evaluated polynomial is needed to compute the number of sign changes in the Sturm sequence, the amount of storage and computation can be reduced, with some loss in accuracy, by scaling and truncating some of least significant bits of the coefficients the $P_k(\omega)$'s. The errors introduced by truncating the coefficients can lead to errors in the sign of the evaluated polynomial. However, this can be controlled by selecting the number of bits truncated. The error due to truncation can be bounded to insure accuracy of the number of sign changes in the Sturm sequence by applying an accuracy test similar to the one described above for checking the accuracy of the polynomial evaluation.

6. SUMMARY

This paper has demonstrated the uniqueness of the unwrapped phase of a one-dimensional finite-length real sequence to within an additive multiple of 2π. It has shown that when the sequence is rational-valued, all-integer arithmetic can be used to exactly compute the number of multiples of π which must be added to the principle value of the phase to give the unwrapped phase. An algorithm for computing the unwrapped phase is provided. This approach should be of interest when highly accurate phase unwrapping is required.

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APPENDIX

The following theorem, not generally found in this form in the literature, is essential in demonstrating the uniqueness of $L(\omega)$. Proof of this theorem is similar to the proof of the Sturm theorem for real polynomials given in [4]. Details of the proof are given in [5].

Theorem 1 The number of sign changes in the real and imaginary parts of a complex polynomial. *Let $F(z)$ be a finite order complex polynomial,*

$$F(z) = \sum_{n=0}^{N} f_n z^n = P_0(z) + jP_1(z)$$

where f_n and z are complex, $F(z)$ has no real zeros in the closed real interval $[a,b]$, and where $P_0(z)$ and $P_1(z)$ are finite order polynomials with real coefficients $p_0(n)$ and $p_1(n)$,

$$\begin{aligned} P_0(z) &= \sum_{n=0}^{N} p_0(n) z^n \\ P_1(z) &= \sum_{n=0}^{N} p_1(n) z^n \end{aligned}$$

with $P_1(z)$ not identically 0. As the point $z = x$ moves along the real axis from the point $x = a$ to $x = b$, let σ be the number of times that $G(x) = P_1(x)/P_0(x)$ changes from $-$ to $+$ at the roots of $P_0(x)$, and τ the number of times that $G(x)$ changes from $+$ to $-$ at the roots of $P_0(x)$ in the interval $[a,b]$. Let $P_2(x)$, $P_3(x)$, ..., $P_S(x)$ be the Sturm sequence formed from $P_0(x)$ and $P_1(x)$ using the negative remainder division algorithm,

$$P_{k-1}(x) = Q_k(x)P_k(x) - P_{k+1}(x)$$

where $Q(x)$ is a polynomial chosen such that the order of the $(k+1)^{th}$ element of the Sturm sequence is less than the order of the k^{th} element until the $P_S(x)$ does not change sign over the interval $[a,b]$. If $P_S(x)$ is identically zero, we assign it an arbitrary constant value. Then,

$$\tau - \sigma = V_a - V_b.$$

where V_a and V_b denote the sign change counting operator applied to the Sturm sequence evaluated at a and b, respectively.

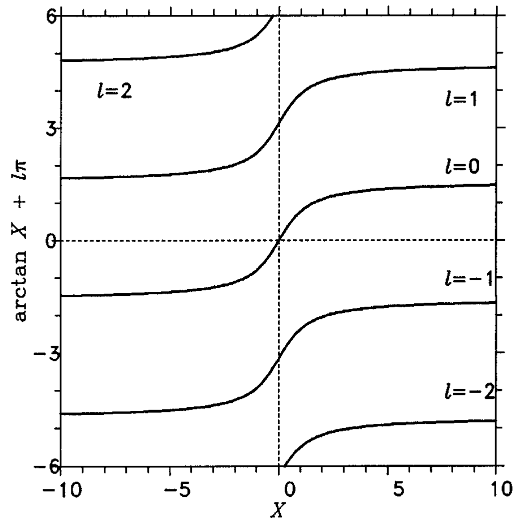

Figure 1: A plot of $arctan x + l\pi$ for several values of l