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Abstract. Sampling plays a crucial role in remote sensing and signal analysis. In classic sampling theory, a signal is sampled at a uniform rate and at a minimum of twice the signal bandwidth. In practice, only finitely many samples of the signal are available and these can often only be obtained at irregular positions due to platform movement, scanning, pulsed operations, etc. In addition, often only local averages of the signal at these irregularly spaced locations can be measured, often with different apertures at different locations. We will use the one-to-one correspondence between the analog and discrete band-limited signal spaces established by the regular sampling theorem, and give a direct treatment of the discrete problem of irregular samplings. This problem is finite-dimensional and linear, easily accessible, and transparent. Its solution can be easily implemented and analyzed to solve engineering applications. We will see that—different from other approaches—only the number of samples, not their location or density, determine whether the signal can be recovered or not.

1. INTRODUCTION. One of the fundamental problems in signal analysis is: Given the samples $f(t_k)$, $k \in \mathbb{Z}$, of a function f in PW^2, under which conditions can f be recovered from its samples? Here PW^2 denotes the Paley–Wiener space

$$\mathrm{PW}^2 := \{f \in L^2(\mathbb{R}) \mid \operatorname{supp} \hat{f} \subseteq [-1/2, 1/2]\}$$

which is often called the space of "band-limited" functions by engineers. Recall that $\hat{f}(\xi) := \int_{\mathbb{R}} f(x)e^{-2\pi i x\xi}\, dx$ is the Fourier transform of f.

If f is band-limited and the samples are equally spaced and sufficiently dense, then it is well known that f can be uniquely recovered by the cardinal series [14, 15].

However, for instance, in a typical remote sensing application, measurements are samples of an aperture-filtered signal. The aperture results from spatial filtering characteristics of the antenna, optics, and/or signal processing used in the signal sampling. Signal sampling may involve a combination of platform movement, scanning, and pulsed operation, etc. Therefore, the samples are often irregularly spaced and constitute only local averages of the signal at the sampling locations, often with different apertures at different locations. If all the samples are obtained with a fixed aperture function, then this function can be removed by deconvolution after the recovery of the signal.

The mathematical approach to the *analog* version of this sampling setting is known as "weighted average sampling." It has become a very active field of research in the last 20 years, where it has been studied in the setting of wavelet spaces, frame theory, etc. See, for example, [1–8, 11].

In this article, we will use the one-to-one correspondence between the analog and discrete band-limited signal spaces established by the regular sampling theorem and deal directly with the discrete problem of irregular sampling, which focuses on reconstruction from a finite number of samples; see [10]. This problem is finite-dimensional and linear and can therefore be handled with methods of linear algebra. It is easily

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accessible, transparent, and its solution can be readily implemented to solve engineering applications. See, for instance, [12].

We show how the problem of reconstructing a discrete band-limited signal from its irregular samples can be easily reduced to the regular case, where it can be directly recovered using the discrete version of the well-known Shannon–Whittaker–Kotel'nikov sampling theorem. We will see that—different from other approaches (see, for instance, [4, 9, 10, 13])—only the number of samples, not their location or density, determines whether the signal can be recovered or not. A modification of this approach allows the reconstruction of discrete band-limited signals from aperture-filtered samples, provided the apertures satisfy some linear independence conditions.

2. PRELIMINARIES. In digital signal processing, a signal consists of a finite amount of data, i.e., it is a finite sequence $\big(f(0), f(1), f(2), \ldots, f(N-1)\big)$ of numbers, where $N \in \mathbb{N}$ can be very large. The *irregular sampling problem* asks the following: Under which conditions can f be completely recovered from its samples $\big(f(n_1), \ldots, f(n_r)\big)$, where the sample locations $n_1, \ldots, n_r \in \mathbb{N}$ with $0 \leq n_1 < n_2 < \ldots, n_r \leq N-1$ are given.

In applications, frequently the more general problem, which we will term the *irregular aperture-filtered sampling problem*, is encountered: Under which conditions can f be completely recovered from its "aperture-filtered" samples $\big(f_{v_1}(n_1), \ldots, f_{v_r}(n_r)\big)$ at the given locations $0 \leq n_1 < \cdots < n_r \leq N-1$; here the samples $f_{v_j}(n_j) = (f * v_j)(n_j)$ (see equation (6)) represent the measured values of f at n_j by some device v_j that provides some "local average" of the values of f in some neighborhood of n_j dependent on the device v_j $(j = 1, \ldots, r)$. Generally, the device used for the measurements may change from sample location to sample location, but we assume that we know the "characteristics" v_j of the measuring devices used.

The Hilbert Space of *N*th-order signals. Since a digital signal in practice has finite length N for some $N \in \mathbb{N}$, it can be modeled by a function f from $\{0, 1, \ldots, N-1\}$ into the complex numbers $\mathbb{C}$. Similar to the theory of analog signals of finite length, we will identify f with its periodic extension. To this end, we identify the set $\{0, 1, \ldots, N-1\}$ with the (finite) cyclic group $\mathbb{Z}_N = \mathbb{Z}/N\mathbb{Z}$ and define a *signal* or, more precisely, a *Nth order signal* to be any (N-periodic) function $f : \mathbb{Z}_N \to \mathbb{C}$. We denote the $\mathbb{C}$-vector space of all Nth order signals by $\ell(\mathbb{Z}_N)$. Clearly, if $f \in \ell(\mathbb{Z}_N)$, then $f(k + mN) = f(k)$ for all $k, m \in \mathbb{N}$.

$\ell(\mathbb{Z}_N)$ is a Hilbert space relative to the inner product

$$\langle f, g \rangle := \sum_{k=0}^{N-1} f(k)\overline{g(k)},$$

which induces the square norm

$$\|f\| = \sqrt{\langle f, f \rangle} = \left(\sum_{k=0}^{N-1} |f(k)|^2 \right)^{1/2}$$

on $\ell(\mathbb{Z}_N)$.

In the following, we introduce some more terminology and notation, which facilitates the presentation and discussion of the results of this article.

Dirichlet Kernel. For convenience, we set

$$W_N := e^{-2\pi i/N}.$$

We list some important, well-known properties of W_N in the following.

Proposition 1. 1. *$W_N^k = 1$ if and only if $k \equiv 0 \pmod N$. In particular, $W_N \neq 1$ for all $N \in \mathbb{N}$.*

2. *For all $k \in \mathbb{Z}$*

$$\sum_{n=0}^{N-1} W_N^{nk} = \begin{cases} N & \text{if } k \equiv 0 \pmod N, \\ 0 & \text{otherwise.} \end{cases}$$

For any $n \in \mathbb{Z}_N$, we define the signals $\mathbf{W}_N^n, \mathbf{e}_n^{(N)} : \mathbb{Z}_N \to \mathbb{C}$ by

$$\begin{aligned} \mathbf{W}_N^n(k) :=& W_N^{nk} = e^{-2\pi i n k/N}, \\ \mathbf{e}_n^{(N)}(k) :=& \delta_{nk} \end{aligned}$$

($k \in \mathbb{Z}_N$) where δ_{nk} denotes the Kronecker delta. The families

$$\begin{aligned} \mathfrak{O}_N :=& \left(N^{-1/2}\mathbf{W}_N^0, \ldots, N^{-1/2}\mathbf{W}_N^{N-1}\right), \\ \mathfrak{E}_N :=& \left(\mathbf{e}_0^{(N)}, \ldots, \mathbf{e}_{N-1}^{(N)}\right) \end{aligned}$$

are orthonormal bases of $\ell(\mathbb{Z}_N)$.

For $M \in \mathbb{N}$ and $M < N/2$, we define the signal $D_M := D_M^{(N)} : \mathbb{Z}_N \to \mathbb{C}$ by

$$D_M := \sum_{k=-M}^{M} \mathbf{W}_N^{-k},$$

which is called the *Dirichlet kernel* of order M and which can be written in closed form as

$$D_M(n) = \begin{cases} \dfrac{\sin\big((2M+1)\pi n/N\big)}{\sin\big(\pi n/N\big)} & \text{if } n \not\equiv 0 \pmod N, \\ 2M+1, & \text{otherwise.} \end{cases} \tag{1}$$

Figure 1 depicts the graphs of the Dirichlet kernel function D_{25} (plotted in bold) and the shifted Dirichlet kernel function $\tau_{10}(D_{25}) = D_{25} * \mathbf{e}_{10}^{(N)}$ in $\ell(\mathbb{Z}_{51})$.

Note that $D_M(0) = 2M + 1$ and $D_M(n) = 0$ if and only if there exists a $k \in \mathbb{Z} \setminus \{0\}$ such that $(2M + 1)n = kN$. If $2M + 1 \mid N$, i.e., if $(2M + 1)d = N$ for some $d \in \mathbb{Z}_N$, then $D_M(n) = 0$ if and only if n is a multiple of d.

Operations. For $n_0 \in \mathbb{Z}_N$, let τ_{n_0}, ϱ and μ_{n_0} denote the operations of *translation* by n_0, *reflection,* and *modulation* by n_0 on $\ell(\mathbb{Z}_N)$, respectively, i.e.,

$$\begin{aligned} \tau_{n_0}(f)(k) :=& f(k - n_0), \\ \varrho(f)(k) :=& f(-k) = f(N - k), \\ \mu_{n_0}(f)(k) :=& \mathbf{W}_N^{n_0}(k) \cdot f(k) \end{aligned}$$

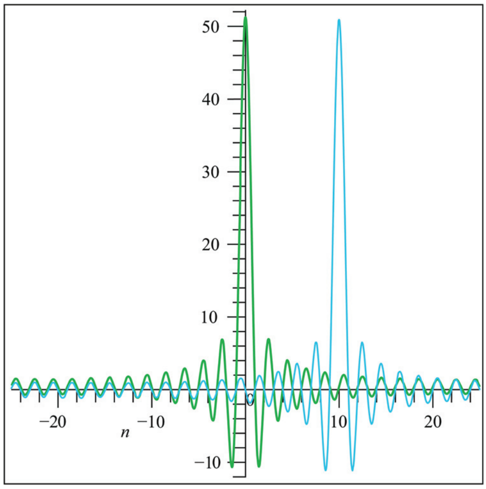

Figure 1. The Dirichlet kernel D_{25} (bold) and $\tau_{10}(D_{25})$ (light line) for $N = 51$.

for $f \in \ell(\mathbb{Z}_N)$ and $k \in \mathbb{Z}_N$. Let $f, g \in \ell(\mathbb{Z}_N)$; then the *convolution* $f * g$ of f and g is defined by

$$(f * g)(k) := \sum_{j \in \mathbb{Z}_N} f(j)g(k - j)$$

for all $k \in \mathbb{Z}_N$.

The discrete Fourier transform. If $f \in \ell(\mathbb{Z}_N)$, then the *discrete Fourier transform* $\hat{f}$ of f is the signal $\hat{f} \in \ell(\mathbb{Z}_N)$ defined by

$$\hat{f} := \frac{1}{\sqrt{N}} \sum_{n=0}^{N-1} f(n)\mathbf{W}_N^n.$$

It can be easily verified that the discrete Fourier transform $\mathcal{F} : \ell(\mathbb{Z}_N) \to \ell(\mathbb{Z}_N) : f \mapsto \hat{f}$ is an isometric isomorphism; in particular, we have Parseval's formula

$$\langle f, g \rangle = \langle \hat{f}, \hat{g} \rangle, \tag{2}$$

and thus

$$\|f\| = \|\hat{f}\| \tag{3}$$

for all $f, g \in \ell(\mathbb{Z}_N)$. Moreover, we have the well-known inversion formula

$$f = \frac{1}{\sqrt{N}} \sum_{n=0}^{N-1} \hat{f}(n)\mathbf{W}_N^{-n}. \tag{4}$$

We list, for easy reference, some further well-known properties of the convolution product and the discrete Fourier transform (see, for instance, [10, 16]).

Proposition 2. *Let $f, g \in \ell(\mathbb{Z}_N)$ and $n, m \in \mathbb{Z}_N$. Then*

1. $\|f * g\| \le \|f\| \|g\|$.
2. $\mathbf{W}_N^n * \mathbf{W}_N^m = \begin{cases} N\mathbf{W}_N^n & \text{if } n \equiv m \pmod N, \\ 0 & \text{otherwise.} \end{cases}$
3. $\tau_n(D_M) * \tau_m(D_M) = N\tau_{n+m}(D_M)$.
4. $\mathcal{F}(N^{-1/2}\mathbf{W}_N^n) = \mathbf{e}_{-n}^{(N)}$.
5. $\mathbf{W}_N^k * \mathbf{e}_\ell^{(N)} = W_N^{-k\ell}\mathbf{W}_N^k$.

Band-limited signals. If $M \in \mathbb{N}$ and $0 < M < N/2$, then the signal $f \in \ell(\mathbb{Z}_N)$ is said to be *M-band-limited* if $\hat{f}(k) = 0$ for all $k \in \mathbb{Z}_N$ with $|k| > M$, i.e., $\operatorname{supp}(\hat{f}) \subseteq \{-M, \ldots, M\}$. For convenience, we will not consider here the case of an asymmetric frequency band. We denote the set of all M-band-limited signals by $\mathcal{B}_M$, i.e.,

$$\mathcal{B}_M = \{f \in \ell(\mathbb{Z}_N) \mid \hat{f}(k) = 0 \text{ for } |k| > M\}.$$

$\mathcal{B}_M = \mathcal{F}^{-1}\big(\langle \mathbf{e}_k^{(N)} \mid |k| \le M \rangle\big)$ constitutes a closed subspace of $\ell(\mathbb{Z}_N)$ of dimension $2M + 1$ with orthonormal basis $\mathfrak{D}_{2M+1} = \big(N^{-1/2}\mathbf{W}_N^k \mid |k| \le M\big)$. We will also use the basis $\tilde{\mathfrak{D}}_{2M+1} = \big(\mathbf{W}_N^k \mid |k| \le M\big)$ of $\mathcal{B}_M$. The *orthogonal projection* P_M of $\ell(\mathbb{Z}_N)$ onto $\mathcal{B}_M$ is given by $\widehat{P_M(f)}(n) = \hat{f}(n)$ for $|n| \le M$ and $\widehat{P_M(f)}(n) = 0$ for $|n| > M$, $f \in \ell(Z_N)$.

Proposition 3. *Let $n, n_1, n_2 \in \mathbb{Z}_N$. Then*

1. $\widehat{D_M} = \sqrt{N}\chi_{\{-M,\ldots,M\}}$. *Moreover,* $\widehat{\tau_n(D_M)} = \mathbf{W}_N^n\sqrt{N}\chi_{\{-M,\ldots,M\}}$, $\tau_n(D_M) \in \mathcal{B}_M$.
2. $\langle \tau_{n_1}(D_M), \tau_{n_2}(D_M) \rangle = ND_M(n_1 - n_2)$. *Hence* $\tau_{n_1}(D_M) \perp \tau_{n_2}(D_M)$ *if and only if* $(n_1 - n_2)(2M + 1) = mN$ *for some* $m \in \mathbb{Z} \setminus \{0\}$.
3. $\|\tau_n(D_M)\| = \sqrt{N(2M + 1)}$.

Proof. (1) The assertions follow immediately from Proposition 2(4):

$$\widehat{D_M} = \mathcal{F}\left(\sum_{k=-M}^{M} \mathbf{W}_N^{-k}\right) = \sum_{k=-M}^{M} \mathcal{F}(\mathbf{W}_N^{-k}) = \sqrt{N} \sum_{k=-M}^{M} \mathbf{e}_{-k}^{(N)} = \sqrt{N}\chi_{\{-M,\ldots,M\}}.$$

(2) Again by Proposition 2 and using the fact that $\mathcal{F}$ preserves the inner product, we can conclude

$$\begin{aligned}
\big\langle \tau_{n_1}(D_M), \tau_{n_2}(D_M) \big\rangle &= \big\langle \widehat{\tau_{n_1}(D_M)}, \widehat{\tau_{n_2}(D_M)} \big\rangle \\
&= \big\langle \mathbf{W}_N^{n_1}\sqrt{N}\chi_{\{-M,\ldots,M\}}, \mathbf{W}_N^{n_2}\sqrt{N}\chi_{\{-M,\ldots,M\}} \big\rangle \\
&= N \sum_{k=0}^{N-1} \mathbf{W}_N^{n_1}(k)\chi_{\{-M,\ldots,M\}}(k) \cdot \mathbf{W}_N^{-n_2}(k)\chi_{\{-M,\ldots,M\}}(k) \\
&= N \sum_{k=-M}^{M} \mathbf{W}_N^k(n_1 - n_2) = ND_M(n_1 - n_2).
\end{aligned}$$

(3) This identity follows immediately from (2). ∎

The discrete regular sampling theorem. Reconstructing a signal from regularly spaced samples is common and can be done using the discrete version of the classical Shannon–Whittaker–Kotel'nikov sampling theorem. For convenience, we provide a formal proof in this article. We first establish a specialized version of the discrete sampling theorem.

Theorem 1. *If $M \in \mathbb{N}$, such that $2M+1$ divides N, i.e., $d(2M+1) = N$ for some $d \in \mathbb{N}$, then:*

1. *For all $j, k \in \{0, 1, \dots, 2M\}$*

$$\tau_{jd}(D_M)(kd) = \begin{cases} 0, & k \neq j; \\ 2M+1, & k = j. \end{cases}$$

2. $\mathfrak{B}_M := \big(\tau_{jd}(D_M) \mid j = 0, \dots, 2M\big)$ *constitutes an orthogonal basis of $\mathcal{B}_M$.*
3. *Any M-band-limited signal $f \in \mathcal{B}_M$ has a basis representation of the form*

$$f = \frac{d}{N} \sum_{j=0}^{2M} f(jd)\tau_{jd}(D_M)$$

relative to $\mathfrak{B}_M$, i.e., $f = \frac{d}{N}\Big(f_s * D_M\Big)$, where $f_s = \sum_{j=0}^{2M} f(jd)\mathbf{e}_{jd}^{(N)} \in \ell(\mathbb{Z}_N)$.

Proof. (1). This assertion follows directly from equation (1) and $(2M+1)d = N$. In fact, if $j, k \in \{0, \dots, 2M\}$, then

$$\tau_{jd}(D_M)(kd) = D_M\big((k-j)d\big) = \begin{cases} \sin\big(\pi(k-j)\big) = 0; & k \neq j \\ 2M+1, & k = j. \end{cases}$$

(2) If $j_1, j_2 \in \{0, 1, \dots, 2M\}$, then $(j_2 d - j_1 d)(2M+1) = (j_2 - j_1)N$, since by assumption $(2M+1)d = N$. Hence, we can conclude using part 2 of Proposition 3 that $\tau_{j_1}(D_M)$ and $\tau_{j_2}(D_M)$ are orthogonal, provided that $j_1 \neq j_2$. As $\dim(\mathcal{B}_M) = 2M+1$, $\mathfrak{B}_M$ is an orthogonal basis of $\mathcal{B}_M$ as claimed.

(3) By (2), any $f \in \mathcal{B}_M$ has a basis expansion

$$f = \sum_{j=0}^{2M} a_j \tau_{jd(D_M)}$$

relative to $\mathfrak{B}_M$, where the coefficients a_j are the Fourier coefficients

$$a_j = \frac{\langle f, \tau_{jd}(D_M)\rangle}{\|\tau_{jd}(D_M)\|^2} = \frac{\langle f, \tau_{jd}(D_M)\rangle}{N(2M+1)}.$$

Thus, we only have to show that $\langle f, \tau_{jd}(D_M)\rangle = Nf(jd)$ for all $j \in \{0, \dots, 2M\}$. Parselval's formula (2) in conjunction with the inversion formula (4) implies

$$\langle f, \tau_{jd}(D_M)\rangle = \langle \hat{f}, \widehat{\tau_{jd}(D_M)}\rangle = \langle \hat{f}, \mathbf{W}_N^{jd}\sqrt{N}\chi_{\{-M,\dots,M\}}\rangle$$

$$= \sqrt{N} \sum_{n=0}^{N-1} \hat{f}(n) \mathbf{W}_N^{-n}(jd) = N f(jd),$$

since, by assumption, $\hat{f}(n) = 0$ for all $|n| > M$. ∎

Corollary 1 (Discrete Regular Sampling Theorem). *Let $M \in \mathbb{N}$ and suppose d is a divisor of N such that $d \leq N/(2M+1)$ and $r := N/d - 1$ is even. Then any M-band-limited signal $f \in \mathcal{B}_M$ has an expansion of the form*

$$f = \frac{d}{N} \sum_{j=0}^{r} f(jd) \tau_{jd}(D_M).$$

Note that the family $\big(\tau_{jd}(D_M) \mid j = 0, \ldots, r\big)$, in general, is neither orthogonal nor linearly independent over $\mathbb{C}$.

Proof. Let $M' := r/2$. Then $N = (2M'+1)d$ and as $d(2M+1) \leq N$, $M \leq M'$ and, therefore, $\mathcal{B}_M \subseteq \mathcal{B}_{M'}$. Thus, if $f \in \mathcal{B}_M$, then $f \in \mathcal{B}_{M'}$ and by Theorem 1, we have the expansion

$$f = \frac{N}{d} \sum_{j=0}^{r} f(jd) \tau_{jd}(D_{M'}) \tag{5}$$

relative to the orthogonal basis $\mathfrak{B}_{M'} = \big(\tau_{jd}(D_{M'}) \mid j = 0, \ldots, 2M' = r\big)$. In order to obtain an expansion in terms of the kernels $\tau_{jd}(D_M)$, we apply the orthogonal projection P_M from $\ell(\mathbb{Z}_M)$ onto $\mathcal{B}_M$ to equation (5), which yields

$$f = P_M(f) = P_M \left(\frac{N}{d} \sum_{j=0}^{r} f(jd) \tau_{jd}(D_{M'}) \right) = \frac{N}{d} \sum_{j=0}^{r} f(jd) P_M\big(\tau_{jd}(D_{M'})\big)$$

and which equals

$$\frac{N}{d} \sum_{j=0}^{r} f(jd) \tau_{jd}(D_M)$$

since $P_M\big(\tau_{jd}(D_{M'})\big) = \tau_{jd}\big(P_M(D_{M'})\big) = \tau_{jd}(D_M)$ as can be easily verified. ∎

3. IRREGULAR SAMPLES. We now turn to the problem of reconstructing a band-limited signal from its irregularly spaced samples.

Let $N, M \in \mathbb{N}$, $(2M+1) \mid N$, and $d := N/(2M+1)$. Moreover, let $r \in \mathbb{N}$, and let $0 \leq n_0 < n_1 < n_2 < \cdots < n_{r-1} \leq N-1$ be the (irregular) sampling locations for the signals in $\mathcal{B}_M$.

Definition 1 (Sampling Homomorphism). We call the unique homomorphism s_M :

$\mathcal{B}_M \to \ell(\mathbb{Z}_r)$ defined by linear extension through the association

$$D_M * \mathbf{e}_{jd}^{(N)} \mapsto \begin{pmatrix} \left(D_M * \mathbf{e}_{jd}^{(N)}\right)(n_0) \\ \left(D_M * \mathbf{e}_{jd}^{(N)}\right)(n_1) \\ \vdots \\ \left(D_M * \mathbf{e}_{jd}^{(N)}\right)(n_{r-1}) \end{pmatrix}$$

($j = 0, 1, \dots, 2M$) the *sampling homomorphism* of $\mathcal{B}_M$ with respect to the locations $n_0, n_1, \dots, n_{r-1}$. The representation matrix S_M for s_M,

$$S_M = M_{\mathfrak{E}_r, \mathfrak{B}_M}(s_M) = \begin{pmatrix} \left(D_M * \mathbf{e}_0^{(N)}\right)(n_0) & \cdots & \left(D_M * \mathbf{e}_{2Md}^{(N)}\right)(n_0) \\ \vdots & & \vdots \\ \left(D_M * \mathbf{e}_0^{(N)}\right)(n_{r-1}) & \cdots & \left(D_M * \mathbf{e}_{2Md}^{(N)}\right)(n_{r-1}) \end{pmatrix},$$

is called the *sampling matrix* of $\mathcal{B}_M$ with respect to $n_0, n_1, \dots, n_{r-1}$ relative to the basis $\mathfrak{B}_M$ and $\mathfrak{E}_{2M+1}$.

Note that the *coordinate isomorphism* $\varphi_{\mathfrak{B}_M} : \mathcal{B}_M \to \ell(\mathbb{Z}_{2M+1})$ of $\mathcal{B}_M$ relative to $\mathfrak{B}_M$, defined by $D_M * \mathbf{e}_j^{(N)} \mapsto \mathbf{e}_j^{(2M+1)}$, $j = 0, 1, \dots, 2M$, by Theorem 1, assigns to each signal $f \in \mathcal{B}_M$ the vector

$$\varphi_{\mathfrak{B}_M}(f) = \tfrac{N}{d}\left(f(0), f(d), \dots, f(2Md)\right)^t,$$

whose components are the values of f at the regular locations $0, d, \dots, 2Md$ scaled by N/d.

The following theorem with its corollary contains the main result of this section. It implies that any M-band-limited function f is uniquely determined as long as we know at least $2M + 1$ samples, independent of how they are spread over its domain.

Theorem 2. *s_M has rank $2M + 1$ if and only if $r \geq 2M + 1$.*

Proof. It suffices to show that $r \geq 2M + 1$ implies $\operatorname{rank}(s_M) = 2M + 1$, since the converse is trivial. Since $\tilde{\mathfrak{H}}_{2M+1} = (\mathbf{W}_N^k \mid k \in \{-M, \dots, M\})$ is a basis of $\mathcal{B}_M$, it suffices to show that

$$M_{\mathfrak{E}_N, \tilde{\mathfrak{H}}_{2M+1}}(s_M) = \begin{pmatrix} W_N^{n_0(-M)} & \cdots & W_N^{n_0 M} \\ \vdots & & \vdots \\ W_N^{n_{r-1}(-M)} & \cdots & W_N^{n_{r-1} M} \end{pmatrix}$$

has rank $2M + 1$. Consider the square submatrix A consisting of the first $2M + 1 \leq r$ rows of $M_{\mathfrak{E}_N, \tilde{\mathfrak{H}}_{2M+1}}(s_M)$. Set $\alpha_j := W_N^{n_j}$ then $W_N^{n_j k} = (\alpha_j)^k$ and

$$A = \operatorname{diag}\left(\alpha_0^{-M}, \dots, \alpha_{2M}^{-M}\right) \cdot V(\alpha_0, \dots, \alpha_{2M}),$$

where the second factor is the Vandermonde matrix

$$V(\alpha_0, \dots, \alpha_{2M}) = \begin{pmatrix} 1 & \alpha_0 & \alpha_0^2 & \cdots & \alpha_0^{2M} \\ 1 & \alpha_1 & \alpha_1^2 & \cdots & \alpha_1^{2M} \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & \alpha_{2M} & \alpha_{2M}^2 & \cdots & \alpha_{2M}^{2M} \end{pmatrix}.$$

Clearly,

$$\det(A) = \prod_{k=0}^{2M} \alpha_k^{-M} \prod_{0\le s<t\le 2M} (\alpha_t - \alpha_s) \neq 0.$$

Indeed, by definition, $\alpha_j = W_N^{n_j} \neq 0$ for all $j = 0, \ldots, 2M$. Moreover, $\alpha_t - \alpha_s = 0$ if and only if $W_N^{n_t - n_s} = 1$ and thus $n_t - n_s \in N\mathbb{Z}$. However, by assumption $0 \le n_0 < n_1 < \cdots < n_{2M} \le N-1$, thus $n_t - n_s \notin N\mathbb{Z}$. It follows that the $2M+1$ columns of A and thus also the columns of $M_{\mathfrak{E}_N, \tilde{\mathfrak{H}}_{2M+1}}(s_M)$ are linearly independent, which concludes the proof. ∎

In practice, the measured values of the M-band-limited signal f at the sampling locations n_j are subject to "noise." The non-M-band-limited component of the noise can be removed by solving the sampling equation $S_M X^f = Y^f$ in the "least-squares sense," which can be done in a numerically stable way using the QR decomposition of S_M.

Corollary 2. *Let $S_M = QR$ be the canonical QR decomposition of S_M into the matrix Q with orthonormal columns and the invertible upper triangular matrix R. If $Y^f = \big(f(n_0), \ldots, f(n_{r-1})\big)^t$ is the sampling vector of a M-band-limited function f, then*

$$X^f := R^{-1} Q^t Y^f = (N/d)\,\big(f(0), f(d), \ldots, f(2Md)\big)^t$$

is the regular sampling vector of f, i.e.,

$$f = \sum_{j=0}^{2M} X_j^f \tau_{jd}(D_M).$$

Proof. Suppose $r \ge 2M+1$ and $S_M \in \mathbb{C}_{r,2M+1}$. By Theorem 2, S_M has linearly independent columns and thus $S_M^t S_M \in \mathbb{C}_{2M+1,2M+1}$ is invertible. Hence, the least-squares solution of $S_M X^f = Y^f$ is given by

$$X^f = (S_M^t S_M)^{-1} S_M^t Y^f$$

which, using the canonical QR-decomposition of S_M, reduces to

$$R^{-1} Q^t Y^f$$

as claimed. ∎

4. APERTURE-FILTERED SAMPLES. In this section, we address the problem of reconstructing a band-limited signal from its "aperture-filtered" samples. Again, let $M, N, r \in \mathbb{N}$, $M < N/2$, $(2M+1)|N$, $d := N/(2M+1)$, and let $\mathfrak{B}_M := \big(D_M * \mathbf{e}_{jd}^{(N)} \mid j = 0, 1, \ldots, 2M\big)$ denote the canonical orthogonal basis of $\mathcal{B}_M$. Moreover, let $\mathfrak{v} := (v_0, \ldots, v_{r-1})$ denote the system of "aperture functions" $v_0, \ldots, v_{r-1} \in \ell(\mathbb{Z}_N)$ representing the response profiles of the devices used for measuring the samples of an M-band-limited function $f \in \mathfrak{B}_M$ at the (irregular) locations $\mathfrak{n} := (n_0, \ldots, n_{r-1})$, $0 \le n_0 < n_1 < \cdots < n_{r-1} \le N-1$. Different from the setting of the previous section, where we assumed that the *exact* values of the M-band-limited signal $f \in \mathfrak{B}_M$ were available at the (irregular) locations $\mathfrak{n}$, we now consider the more general situation where at each sampling location n_k we only know the value of f convolved with the "aperture function" v_k for $k = 0, \ldots, r-1$. We therefore only know the values

$$(f * v_0)(n_0), \ldots, (f * v_{r-1})(n_{r-1}) \tag{6}$$

of the M-band-limited function f. If the sampling values are measured using the same device at each sampling location, i.e., if $v_0 = \cdots = v_{r-1} = v$, then f can be easily recovered provided $\operatorname{supp}(\hat{v}) \subset [-M, M]$. In the general case, we can also recover f if the aperture functions $v_0, \ldots, v_{r-1}$ satisfy some linear independence conditions.

Definition 2 (Sampling Homomorphism). As above, we call the unique homomorphism $s_M^{\mathfrak{v}} : \mathcal{B}_M \to \ell(\mathbb{Z}_r)$ defined by linear extension through the association

$$D_M * \mathbf{e}_{jd}^{(N)} \mapsto \begin{pmatrix} \left(D_M * \mathbf{e}_{jd}^{(N)} * v_0\right)(n_0) \\ \left(D_M * \mathbf{e}_{jd}^{(N)} * v_1\right)(n_1) \\ \vdots \\ \left(D_M * \mathbf{e}_{jd}^{(N)} * v_{r-1}\right)(n_{r-1}) \end{pmatrix}$$

($j = 0, \ldots, 2M$) the *sampling homomorphism of $\mathcal{B}_M$ with respect to* $\mathfrak{n}$ *relative to the apertures* $\mathfrak{v}$. The representation matrix

$$\begin{aligned} S_M^{\mathfrak{v}} &= M_{\mathfrak{E}_r, \mathfrak{B}_M}\left(s_M^{\mathfrak{v}}\right) \\ &= \begin{pmatrix} \left(D_M * \mathbf{e}_0^{(N)} * v_0\right)(n_0) & \cdots & \left(D_M * \mathbf{e}_{2Md}^{(N)} * v_0\right)(n_0) \\ \vdots & & \vdots \\ \left(D_M * \mathbf{e}_0^{(N)} * v_{r-1}\right)(n_{r-1}) & \cdots & \left(D_M * \mathbf{e}_{2Md}^{(N)} * v_{r-1}\right)(n_{r-1}) \end{pmatrix} \end{aligned}$$

is called the *sampling matrix of $\mathcal{B}_M$ with respect to* $\mathfrak{n}$ *relative to the apertures* $\mathfrak{v}$. Moreover, let $\mathcal{B}_M(\mathfrak{v})$ denote the subspace of all M-band-limited signals $f \in \mathcal{B}_M$ that can be reconstructed from their samples at $\mathfrak{n}$ relative to the apertures $\mathfrak{v}$.

The M-band-limited signal f can be completely recovered from its r aperture-filtered samples if the conditions of the following theorem are satisfied. Note that $\operatorname{rank}(S_M^{\mathfrak{v}}) = 2M + 1$ implies that $r \geq 2M + 1$. Again, the non-M-band-limited noise component is eliminated by the least-squares solution used.

Theorem 3. *Suppose $S_M^{\mathfrak{v}}$ has rank $2M + 1$ and $S_M^{\mathfrak{v}} = QR$ is the canonical QR-decomposition of $S_M^{\mathfrak{v}}$. If $Y^f = \left((f * v_0)(n_0), \ldots, (f * v_{2M})(n_{r-1})\right)^t$ is the vector of the aperture-filtered samples of the M-band-limited signal f, then $X^f = R^{-1} Q^t Y^f$ is the regular sampling vector of f, i.e.,*

$$f = \sum_{j=0}^{2M} X_j^f \tau_{jd}(D_M).$$

We will now investigate under which conditions on the aperture functions $(v_0, \ldots, v_{r-1})$ the sampling matrix has linearly independent columns. We begin with the special case that all samples are taken using the same aperture v assuming that $r \geq 2M + 1$.

Theorem 4. *Suppose $\mathfrak{v} = (v, \ldots, v)$ and $r \geq 2M + 1$. Then $S_M^{\mathfrak{v}}$ has rank $2M + 1$ if and only if $\hat{v}(n) \neq 0$ for all $|n| \leq M$.*

Proof. Consider the endomorphism $c_v : \mathcal{B}_M \to \mathcal{B}_M$ defined by $f \mapsto f * v$. Then $s_M^{\mathfrak{v}} = s_M \circ c_v$ and $\operatorname{rank}(s_M^{\mathfrak{v}}) = \operatorname{rank}(c_v)$ since $r \geq 2M+1$ implies $\ker(s_M) = \{0\}$. However, $\operatorname{rank}(c_v) = 2M+1$ if and only if $\ker(c_v) = \{0\}$, which is equivalent to $\hat{v}(n) \neq 0$ for all $|n| \leq M$. ■

The following theorem gives more insight in the case of uniform aperture functions.

Theorem 5. *Suppose $r \geq 2M+1$, $\mathfrak{v} = (v, \dots, v)$ and the aperture v has the basis representation*

$$v = \sum_{\ell \in \mathbb{Z}_N} a_\ell N^{-1/2} \mathbf{W}_N^\ell$$

relative to the orthonormal basis $\mathfrak{O}_N = (N^{-1/2}\mathbf{W}_N^\ell \mid \ell \in \mathbb{Z}_N)$. Then

$$S_M^{\mathfrak{v}} = N^{1/2} S_M \operatorname{diag}(a_{-M}, \dots, a_M).$$

In particular, $\operatorname{rank}(S_M^{\mathfrak{v}}) = 2M+1$ *provided that $a_n \neq 0$ for all $|n| \leq M$.*

Proof. The identity follows easily from the equations $s_M^{\mathfrak{v}} = s_M \circ c_v$ and $S_M^{\mathfrak{v}} = M_{\mathfrak{E}_r, \mathfrak{B}_M}(s_M) M_{\mathfrak{B}_M, \mathfrak{O}_{2M+1}}(\mathrm{id}) M_{\mathfrak{O}_{2M+1}, \mathfrak{B}_M}(c_v)$. Since, by parts 2 and 5 of Proposition 2,

$$\begin{aligned}
c_v\big(D_M * \mathbf{e}_{jd}^{(N)}\big) &= D_M * \mathbf{e}_{jd}^{(N)} * v \\
&= \left(\sum_{k=-M}^{M} \mathbf{W}_N^{-k}\right) * \left(\sum_{\ell \in \mathbb{Z}_N} a_\ell N^{-1/2} \mathbf{W}_N^\ell\right) * \mathbf{e}_{jd}^{(N)} \\
&= \left(\sum_{k=-M}^{M} a_k \sqrt{N} \mathbf{W}_N^k\right) * \mathbf{e}_{jd}^{(N)} \\
&= \sum_{k=-M}^{M} \sqrt{N} a_k W_N^{-jdk} \mathbf{W}_N^k,
\end{aligned}$$

the representation matrix of c_v relative to the basis $\mathfrak{B}_M$ and $\mathfrak{O}_{2M+1}$ is

$$M_{\mathfrak{O}_{2M+1}, \mathfrak{B}_M}(c_v) = \left(\sqrt{N} a_k W_N^{-jdk}\right)_{k,j} = \sqrt{N} H \operatorname{diag}(a_{-M}, \dots, a_M),$$

where

$$H = \begin{pmatrix}
1 & W_N^{Md} & W_N^{M(2d)} & \cdots & W_N^{M2Md} \\
1 & W_N^{(M-1)d} & W_N^{(M-1)(2d)} & \cdots & W_N^{(M-1)2Md} \\
\vdots & \vdots & \vdots & \vdots & \vdots \\
1 & W_N^{(-M)d} & W_N^{(-M)(2d)} & \cdots & W_N^{(-M)2Md}
\end{pmatrix}.$$

Clearly, $M_{\mathfrak{O}_{2M+1}, \mathfrak{B}_M}(\mathrm{id}) = H$ and therefore $M_{\mathfrak{B}_M, \mathfrak{O}_{2M+1}}(\mathrm{id}) = H^{-1}$. It follows that $S_M^{\mathfrak{v}} = S_M H^{-1} \sqrt{N} H \operatorname{diag}(a_{-M}, \dots, a_M) = S_M \sqrt{N} \operatorname{diag}(a_{-M}, \dots, a_M)$. Since the Fourier transform of v is

$$\mathcal{F}(v) = \mathcal{F}\left(\sum_{\ell \in \mathbb{Z}_N} a_\ell N^{-1/2} \mathbf{W}_N^\ell\right) = \sum_{\ell \in \mathbb{Z}_N} a_\ell \mathbf{e}_{-\ell}^{(N)},$$

it follows immediately that $\hat{v}(n) = a_{-n}$ for $n \in \mathbb{Z}_N$ and the claim follows from the previous theorem. ■

Remark 1. Note that

1. $Y \in \ell(\mathbb{Z}_N)$ is the $\mathfrak{v}$-aperture-filtered sample of a M-band-limited signal if and only if $Y \in \operatorname{im}(s_M^{\mathfrak{v}})$.
2. The band-limited signals $f_1, f_2 \in \mathcal{B}_M$ have the same $\mathfrak{v}$-aperture-filtered samples if and only if $f_2 - f_1 \in \ker(s_M^{\mathfrak{v}})$.
3. $\mathcal{B}_M(\mathfrak{v}) \cong \mathcal{B}_M / \ker(s_M^{\mathfrak{v}})$. In particular, $\dim \mathcal{B}_M(\mathfrak{v}) = \operatorname{rank}(s_M^{\mathfrak{v}})$.

Note: If $\ker(s_M^{\mathfrak{v}}) \neq \{0\}$, then $\mathcal{B}_M(\mathfrak{v})$ is a proper subspace of $\mathcal{B}_M$. $\mathcal{B}_M(\mathfrak{v})$ is not necessarily of the form $\mathcal{B}_{M'}$ for some $0 \leq M' \leq M$.

The following theorem characterizes those systems of aperture functions $\mathfrak{v}$ for which the sampling homomorphism has full rank. We will see that any M-band-limited signal f can be completely recovered from its aperture-filtered samples $\big((f * v_0)(n_0), \dots, (f * v_{2M})(n_{r-1})\big)$ if and only if the family of "shifted apertures" $\big(\tau_{n_k}(v_k) \mid k = 0, \dots, 2M\big)$ is linearly independent "over the frequency band $\{-M, \dots, M\}$," i.e., if the family $\big(P_M(v_k * \mathbf{e}_k^{(N)}) \mid k = 0, \dots, 2M\big)$ is linearly independent.

Theorem 6. *The following statements are equivalent.*

1. $s_M^{\mathfrak{v}}$ *has rank* $2M + 1$.
2. *There exist* $2M + 1$ *indices* $k_0, \dots, k_{2M}$ *in* $\{0, 1, 2, \dots, r - 1\}$ *such that the system* $\big(\hat{v_{k_j}} \mathbf{W}_N^{n_{k_j}} \chi_{\{-M,\dots,M\}} \mid j = 0, \dots, 2M\big)$ *is linearly independent.*
3. *There exist* $2M + 1$ *indices* $k_0, \dots, k_{2M}$ *in* $\{0, 1, 2, \dots, r - 1\}$ *such that the system* $\Big(P_M\big(v_{k_j} * \mathbf{e}_{n_{k_j}}^{(N)}\big) \mid k = 0, \dots, 2M\Big)$ *is linearly independent.*

Proof. We only have to establish the equivalence between (1) and (2). As the signals $D_M * v_k * \mathbf{e}_{n_k}^{(N)}$ are M-band-limited, by Corollary 2 the system

$$\Big(\big(S_M^{\mathfrak{v}}\big)_k \,\Big|\, k = 0, \dots, r - 1\Big) \tag{7}$$

of all row vectors of $S_M^{\mathfrak{v}}$

$$\begin{aligned}\big(S_M^{\mathfrak{v}}\big)_k &= \Big(\big(D_M * \mathbf{e}_0^{(N)} * v_k\big)(n_k) \quad \cdots \quad \big(D_M * \mathbf{e}_{2Md}^{(N)} * v_k\big)(n_k)\Big)\\ &= \Big((D_M * v_k * \mathbf{e}_{-n_k}^{(N)})(0) \quad \cdots \quad (D_M * v_k * \mathbf{e}_{-n_k}^{(N)})(-2Md)\Big)\end{aligned}$$

has rank $2M + 1$ if and only if there exist indices $k_0, \dots, k_{2M}$ in $\{0, \dots, r - 1\}$ such that the system $\big(D_M * v_k * \mathbf{e}_{n_k}^{(N)} \mid j = 0, 1, \dots, 2M\big)$ is linearly independent, which, by applying the discrete Fourier transform, is equivalent to the linear independence of the system $\big(\chi_{\{-M,\dots,M\}} \hat{v}_{k_j} \mathbf{W}_N^{n_{k_j}} \mid j = 0, \dots, 2M\big)$, which establishes the assertion. ■

5. NUMERICAL EXAMPLES. To illustrate the reconstruction technique introduced in this article, we provide a simple numerical example with $M = 2$, $d = 3$, and $N = 15$. A M-band-limited simulated real signal f (shown in Figure 2) with the spectrum $\hat{f}$ shown in Figure 3 was generated. $\hat{f}$ for $k = -2, -1, 0, 1, 2$ is $9, -9, 5, -9, 9$. Arbitrarily chosen irregular samples are at $n = \{2, 3, 4, 6, 13\}$. The sampling matrix

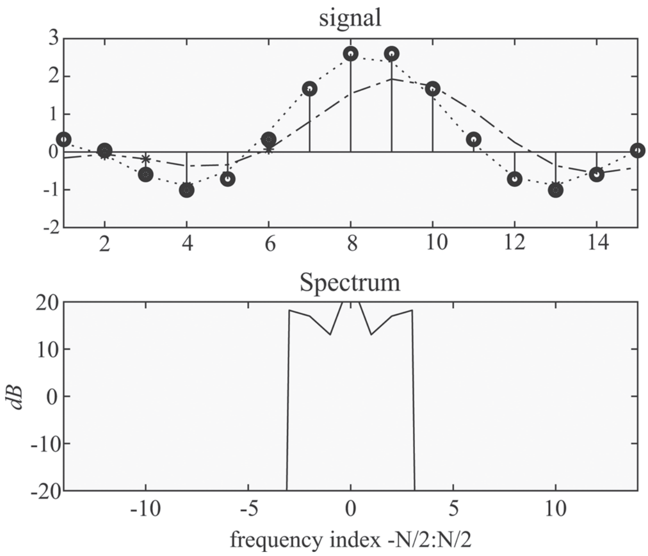

Figure 2. Discrete signal f example shown as stem plot. Filled circles correspond to non-aperture-filtered irregular samples. Dotted and dot-dashed lines show aperture filtered signals. Asterisks show variable aperture-filtered samples.

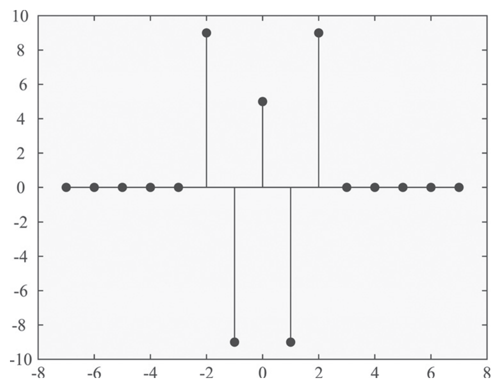

Figure 3. Spectrum $\hat{f}$ of signal f. For simplicity, both are chosen to be real in this example.

S_M and its inverse corresponding to the irregular sampling locations are illustrated in Figure 4. The condition number for the inverse is approximately 40. The result of applying the inverse sampling map to the signal is illustrated in Figure 5. The reconstruction is perfect to within numerical computation accuracy (magnitude error between reconstructed f and f less than 1×10^{-14}).

For the aperture-filtered case, two different aperture functions are defined in Figure 6. The results of convolving each aperture functions and the signal are shown in Figure 2. The sampling alternately selects the value from each aperture-filtered signal at the irregular sample locations illustrated in Figure 2. The corresponding sampling map with respect to the aperture functions and its inverse are shown in Figure 7. The condition number of the map is approximately 37 for this case. The reconstruction is identical to Figure 5. The reconstruction is perfect to within numerical computation accuracy (magnitude error between reconstructed f and f less than 1×10^{-13}).

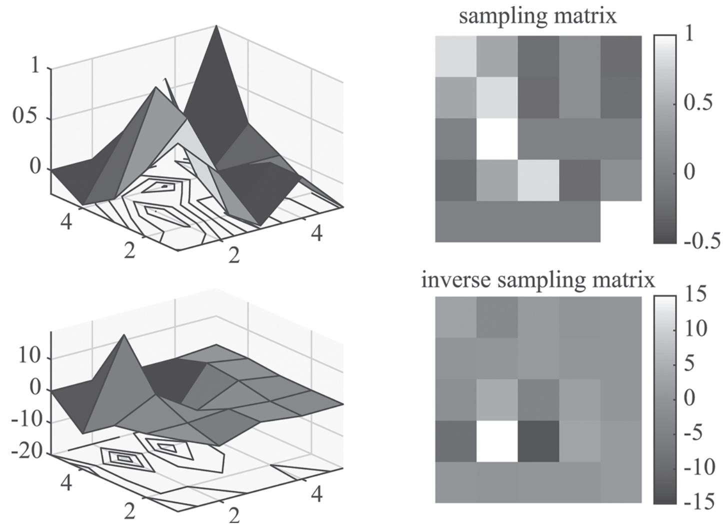

Figure 4. Sampling matrix S_M and its inverse for irregular samples at $n = \{2, 3, 4, 6, 13\}$.

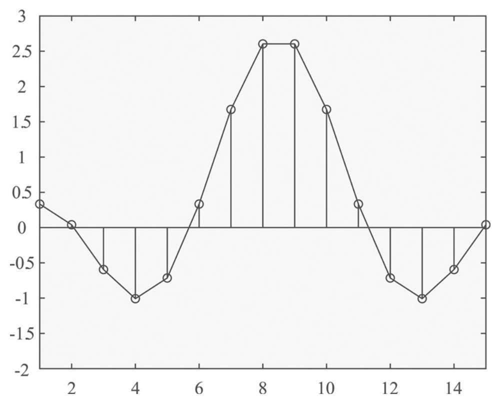

Figure 5. Reconstructed signal (stem plot) compared to the original signal (line).

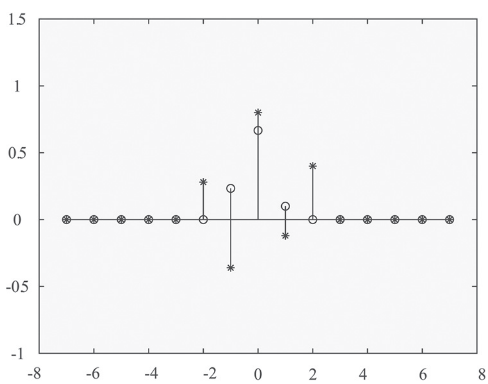

Figure 6. Plot of the two arbitrary aperture functions used. One is shown as open circles, the other as asterisks.

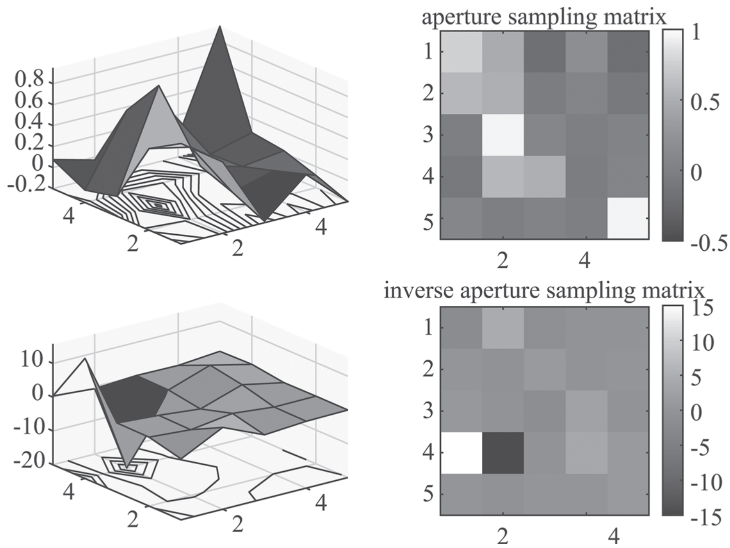

Figure 7. Sampling map S_M^v with respect to v for irregular samples and its inverse.

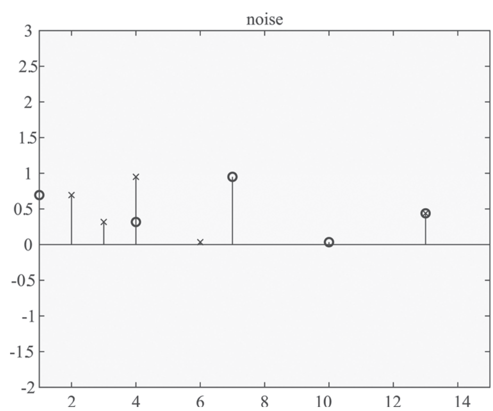

Figure 8. Random values (noise) added to samples. Dark x's are the noise values added to the irregular samples, while circles are the values added to the uniform samples.

To illustrate the effects of noise in measurements of samples, an independent identically distributed (i.i.d.) Gaussian random sequence was generated with variance 1/400. This was added to the measurements for each case. Figure 8 shows the i.i.d. noise values added to the samples (shown Figure 2). Applying the reconstruction matrices to the noise values, Figure 9 shows the results, which demonstrates that the error due to noisy measurements is unevenly distributed, with larger error further from sample locations. The noisy reconstructions are shown in Figure 10. The root mean sum (rms) error for each case is shown in Table 1.

We note that the condition number of the reconstruction matrices depends on the locations of the samples. For uniform sampling the matrix condition number is 1. For the nonideal irregular samples (which are bunched up on the left, see Figure 2) the condition number is approximately 40. For the variable aperture, irregular sampling case, the condition number is approximately 37.

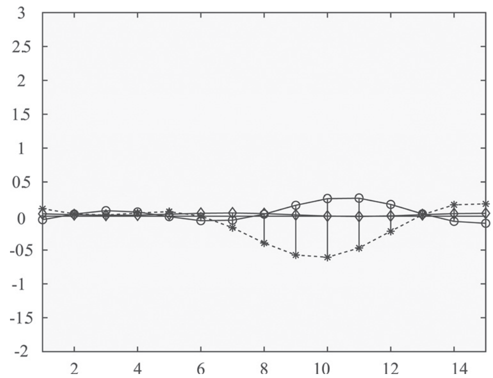

Figure 9. Errors due to random noise added to samples. For the same errors, the uniform error (diamonds) is small and uniformly spread. The error is larger when the sampling is irregular, especially for locations distance from sample locations. The error in the irregular, constant aperture noisy reconstruction case is shown in circles with light bars. The asterisks with dark lines show the irregular, variable aperture noisy reconstruction error.

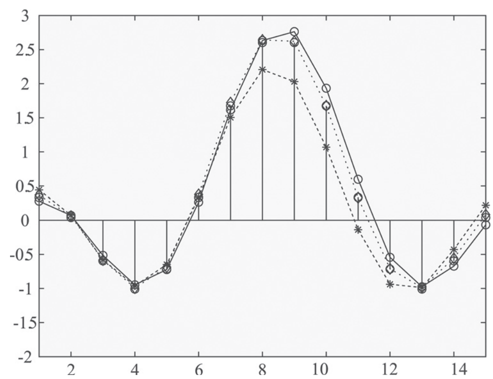

Figure 10. Noisy reconstructed signals compared to the original signals. The original signal is shown with circles. The reconstruction from uniform samples is shown as the light dotted line, while the irregularly sampled reconstructed signal with a constant aperture and added noise is the dark dotted line with asterisks and the irregularly sampled reconstructed signal with variable aperture and added noise is the thin line with circles.

Table 1. rms error for noisy reconstruction.

Case	**rms error**
Samples alone	0.336
Uniform	0.00084
Irregular fixed aperture	0.082316
Irregular variable aperture	0.0155

6. CONCLUSION. In order to keep the presentation focused, we have not considered the issues of over-sampling or noise in our discussion of reconstructing band-limited functions from irregular (aperture-filtered) samples. The standard method for utilizing the additional sampling information in regular sampling is to use the pseudo-inverse of the sampling matrix. This can be extended to the case of irregular sampling.

Moreover, it might be worthwhile to develop a direct algorithm for computing the inverse of the sampling matrix S_M based on the inversion formula for Vandermonde matrices to speed up the reconstruction process. The results can be extended to higher dimensions; see, for instance, [12].

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