Fourier Analysis and Other Tools for Electrical Engineers: A Practical Handbook

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Contents

Forward

This book is intended to provide useful resource for information about Fourier Analysis and related transforms. While many excellent texts have been written on the subject, this book is intended to be a working reference designed especially for Electrical Engineers using common notation and definitions in the field. Key concepts and applications of Fourier Analysis are outlined with the focus on the approaches encountered by Electrical Engineers in practice. This is followed by comprehensive transform and auxilary reference tables. For detailed derivations and explanations the reader is referred to books, for example, those by Bracewell [4], Elliott and Rao [8], Oppenheim and Schafer [18], Papoulis [19], Poularikas [20] and Ulaby and Yagle [24], among others.

The outline of this book is as follows: First, signal fundamentals, notational conventions, the definition of the Fourier Transform, are given in Chapter 1. In Chapters 2 and 3, the Fourier transform family and its relations are derived from each other from different view points. Presenting multiple viewpoints can significantly enhance a student's understanding of the Fourier Analysis: In Chapter 2, the Fourier Series is introduced as a special case of an orthogonal transform. In Chapter 3 the Fourier transform is derived from the Fourier Series and the Fourier Series is derived from the Fourier Transform. The Fourier Transform as the limit of finite interval integral is also explored. The Discrete Time Fourier transform and the Discrete Fourier Transform are discussed in Chapter 4. In Chapter 5, the relationship of the Fourier transform and the LaPlace and $\mathcal{Z}$ transforms is explored. Related transforms are introduced in Chapter 6 and applications of Fourier Analysis common to electrical engineering are discussed in Chapter 7. Chapter 8 consists of comprehensive transform tables for the Fourier transform, Fourier series, and related transforms. Chapter 9 contains useful tables of trigonmetric identities, series, physical constants, unit conversions, etc. An extensive index is provided.

This reference is not complete, and probably never can be! However, it is released with the hope that it can be a useful aid to students in Electrical Engineering. Your comments and suggestions are solicted.Please send your suggestions to the author at `long@ee.byu.edu`.

Notation and Glossary

Note: in general,
$x(t)$ denotes a continuous time function (signal)
$x[n]$ denotes a discrete sequence (signal)
$F\{\cdot\}$ denotes an operation by F
$x * y$ denote convolution of the signals x and y

$\delta[n]$:	Discrete delta function.
$\delta(t)$:	Dirac delta function.
$\delta'(t)$:	Derivative of the Dirac delta function (doublet).
$E\{\cdot\}$:	Statistical expectation.
$Even\{\cdot\}$:	Even part of a function.
f:	Frequency (Hz or 1/s).
$f(t)$, $g(t)$:	General signal functions.
$F(\omega)$:	Fourier transform of $f(t)$.
$F(s)$:	Laplace transform of $f(t)$.
$F(z)$:	$\mathcal{Z}$ transform of $f(t)$.
$\mathcal{F}\{\cdot\}$:	Forward Fourier tranform.
$\mathcal{F}^{-1}\{\cdot\}$:	Inverse Fourier tranform.
$\Gamma(x)$:	Gamma function.
$\mathcal{H}\{\cdot\}$:	Hilbert tranform.
$\mathcal{H}_p\{\cdot\}$:	Hartley tranform.
$\mathcal{H}_n\{\cdot\}$:	Hankel tranform.
$Im\{\cdot\}$:	Imaginary part.
$\mathcal{I}$:	Continuous signal definition interval.
$\mathcal{I}_d$:	Discrete signal definition interval.
j:	The imaginary number $j = \sqrt{-1}$.
k, n, l, m:	Sample index numbers (unitless).
K_F:	Forward Fourier transform scale factor.

K_I:	Inverse Fourier transform scale factor.
$\mathcal{L}\{\cdot\}$:	Forward bilateral (two-sided) Laplace tranform.
$\mathcal{L}^{-1}\{\cdot\}$:	Inverse bilateral (two-sided) Laplace tranform.
$\mathcal{L}_+\{\cdot\}$:	Forward unilateral (one-sided) zero-positive Laplace tranform.
$\mathcal{L}_-\{\cdot\}$:	Forward unilateral (one-sided) zero-negative Laplace tranform.
$\mathcal{L}_+^{-1}\{\cdot\}$:	Forward unilateral (one-sided) zero-positive Laplace tranform.
$\mathcal{L}_-^{-1}\{\cdot\}$:	Forward unilateral (one-sided) zero-negative Laplace tranform.
$\mathcal{L}_*\{\cdot\}$:	Forward discrete Laplace transform (starred Laplace transform.
$\mathcal{L}_*^{-1}\{\cdot\}$:	Inverse discrete Laplace transform (inverse starred Laplace transform.
N:	Discrete signal period (unitless).
$Odd\{\cdot\}$:	Odd part of a function.
$\Pi(t)$:	Gate or rect function.
ω:	Angular radian frequency $\omega = 2\pi f$ (radians/s).
ω_0:	Fundamental frequency (radians/s).
Ω:	Angular frequency (radians/s) (associated with discrete signals).
$Re\{\cdot\}$:	Real part.
s:	Laplace transform variable $s = \sigma + j\omega$.
$S_T(t)$:	Sampling or picket fence function.
sgn(t):	Signum (or, sign) function.
σ:	real part of the Laplace transform variable $s = \sigma + j\omega$.
$\coprod(t)$:	“Shah” or sampling function.
t:	Time (often in s).
T:	Sample interval or Period (in s).
T_0:	Period (often in s).
T_n:	n^{th} order Chebychev polynomial of the first kind.
τ:	Time (often in s).
$u(t)$:	Unit step function.
$u[n]$:	Discrete unit step function.
U_n:	n^{th} order Chebychev polynomial of the second kind.
$\Lambda(t)$:	Triangle function.
λ:	spatial radian frequency (2π/distance).
z:	$\mathcal{Z}$ transform variable $z = re^{j\omega}$.
$\zeta(x)$:	Riemann’s zeta function.

$\mathcal{Z}\{\cdot\}$:	Forward $\mathcal{Z}$ tranform.
$\mathcal{Z}^{-1}\{\cdot\}$:	Inverse $\mathcal{Z}$ tranform.
$\langle\cdot\rangle$:	Statistical average.
A^*:	Complex conjugate of A.
$x(t) * y(t)$:	Convolution of $x(t)$ and $y(t)$.

Chapter 1

Fundamentals and Introduction to the Fourier Transform

1.1 Continuous and Discrete Signals

Following standard conventions, a *continuous signal* $x(t)$ can be viewed as a function of a continuous parameter t over a specified interval $t \in \mathcal{I} = [a, b]$ (generally $\mathcal{I} = [-\infty, \infty]$ unless otherwise specified). Note that $x(t)$ is allowed to have discontinuities or be infinite at some values of t. Generally, t is time but it may also be distance or some other parameter. A two-dimensional continuous signal can be written as $x(t_1, t_2)$. $x(t)$ and $x(t_1, t_2)$ are generally real, but may be complex. If $x(t)$ is a complex signal then we may write,

$$x(t) = x_r(t) + jx_i(t)$$

where $x_r(t)$ and $x_i(t)$ are strictly real and $j = \sqrt{-1}$ is the imaginary number[1].

A *discrete-time signal* or discrete signal $x[n]$ can be viewed as a function of the discrete parameter n where n is a member of the set of integers, i.e., $n \in \mathcal{I}_d = \{n\}$. The square brackets holding the argument distinquish discrete functions from continuous functions. Generally $\mathcal{I}_d$ is the set of *natural numbers* (the set of all integers both positive and negative, including zero). Discrete signals are frequently called *sequences*. A two-dimensional discrete signal (sequence) is written as, $x[n_1, n_2]$. As in continuous signals, discrete signals may be real or complex. Two commonly used relationships between a discrete $x[n]$ and a continuous signal $x(t)$ are,

$$\begin{aligned} x[n] &= \frac{1}{T} x(nT) \\ x[n] &= x(nT) \end{aligned}$$

where T is the sample period. For the discrete real signal to exactly represent the continuous signal, the sample rate $1/T$ must be twice the highest frequency present in $x(t)$. This is the Nyquist frequency. The Nyquist frequency for a complex signal is highest frequency present. *Sampling*, or the conversion of a continuous signal to a discrete signal, is discussed further in Chapter 8.

[1]Electrical Engineers generally prefer to use j rather than i for the imaginary number.

1.2 Even and Odd Signals

Note that a continuous signal $x(t)$ can be written in terms of its **even** and **odd** components as,

$$x(t) = x_e(t) + x_o(t)$$

where

$$\begin{aligned} x_e(t) &= \frac{1}{2}\left[x(t) + x(-t)\right] = Re\{x(t)\} \\ x_o(t) &= \frac{1}{2}\left[x(t) - x(-t)\right] = Im\{x(t)\}. \end{aligned}$$

Similarly, a discrete signal $x[n]$ can be written in terms of its **even** and **odd** components as,

$$x[n] = x_e[n] + x_o[n]$$

where

$$\begin{aligned} x_e[n] &= \frac{1}{2}\left(x[n] + x[-n]\right) = Re\{x[t]\} \\ x_o[n] &= \frac{1}{2}\left(x[n] - x[-n]\right) = Im\{x[t]\}. \end{aligned}$$

A complex signal is termed Hermitian if the real part is even and the imaginary part is odd. It is anti-Hermitian if the real part is odd and the imaginary part is odd.

1.3 Shifting and Scaling

A signal $x(t)$ is time shifted by a value of a by writing $x(t-a)$. If a is positive, the time shift can be viewed as a delay, i.e., $x(t-a)$ is to the right of $x(t)$. For negative a the time is advanced so that $x(t-a)$ is left of $x(t)$.

A time scaled signal is written as $x(at)$. For $a > 1$, $x(at)$ is compress relative to $x(t)$ while for $0 < a < 1$, $x(t)$ is expanded or lengthened relative to $x(t)$. For a negative, $x(at)$ is time-reversed of flipped about the origin.

1.4 Causality and Stability

A continuous signal is *causal* if $x(t) = 0$ for $t < 0$. It is termed *anti-causal* if $x(t) = 0$ for $t > 0$. For discrete signals (sequences) the sequence is causal if $x[n] = 0$ for $n < 0$ and anti-causal if $x[n] = 0$ for $n > 0$.

A continuous signal is bounded if $|x(t)| < K$ for all t where K is some constant. By definition, all discrete signals are bounded, i.e., $|x[n]| < K$. A discrete signal (sequence) corresponding to the impulse response of a system is called *stable* if and only if it is *absolutely summable*, i.e., if and only if

$$\sum_{n=-\infty}^{\infty} |x[n]| < \infty.$$

A continuous system is *stable* if its impulse response is *absolutely summable*,

$$\int_{-\infty}^{\infty} |x(t)| dt < \infty.$$

1.5 Convolution and Correlation

Convolution of two continuous signals $f(t)$ and $g(t)$ is denoted by $f(t) * g(t)$ (or, $f * g$ to be short) and is defined as,

$$f(t) * g(t) = \int_{-\infty}^{\infty} f(\tau)g(t-\tau)d\tau = \int_{-\infty}^{\infty} f(t-\tau)g(\tau)d\tau$$

since convolution is commutative. Note that convolution operations can be viewed as the integration of the product of a signal and a time-shifted signal.

For discrete signals, convolution is defined as,

$$f[n] * g[n] = \sum_{k=-\infty}^{\infty} f[k]g[n-k] = \sum_{k=-\infty}^{\infty} f[n-k]g[k].$$

The **time correlation function** $\mathcal{R}_{fg}(\tau)$ of two signals $f(t)$ and $g(t)$ is defined as,

$$\mathcal{R}_{fg}(\tau) = \int_{-\infty}^{\infty} f(t)g(t+\tau)dt = \int_{-\infty}^{\infty} f(t)g(t-\tau)dt = \mathcal{R}_{fg}(-\tau).$$

A special case of the correlation function is the **time autocorelation function** $\mathcal{R}_g(\tau)$

$$\mathcal{R}_g(\tau) = \int_{-\infty}^{\infty} g(t)g(t+\tau)dt = \int_{-\infty}^{\infty} g(t)g(t-\tau)dt.$$

For random processes, the **correlation function** is the expected value of $f(t)g(t+\tau)$, i.e., $\mathcal{R}_{fg}(t,\tau) = E\{f(t)g(t+\tau)\}$. When both f and g are *wide sense stationary* then $\mathcal{R}_{fg}(t,\tau) = \mathcal{R}_{fg}(t-\tau)$.

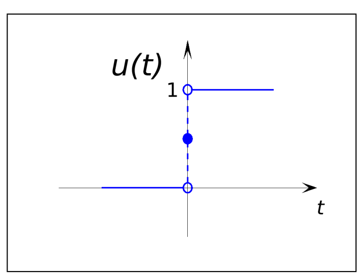
1.6 Ordinary Functions

In this section some symbols for commonly used ordinary functions are defined in single and multiple dimensions.

1.6.1 Continuous Functions

The **Continuous Unit Step** function denoted by $u(t)$ is

$$u(t) = \begin{cases} 1 & t > 0 \\ (1/2 & t = 0) \\ 0 & t < 0. \end{cases}$$



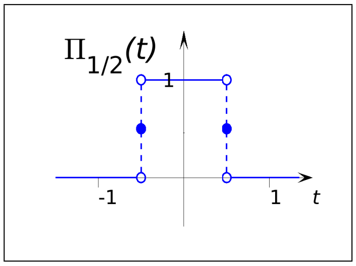
Often, the $u(t)$ is defined to be 1 at $t = 0$ rather than 1/2. For most applications this distinction is irrelevant. The definition given here simplifies computation of the derivative. $u(t)$ is also known as the *Heavyside function* that may be denoted by $H(t)$.

In multiple dimensions the N dimensional **Multi-D Continuous Unit Step** function is[2]

$$u(x_1, x_2, \ldots, x_N) = u(x_1)u(x_2)\cdots u(x_N).$$

The **Continuous Unit Gate** function (also known as the *rect* function) denoted here by $\Pi(t)$ is

$$\Pi(t) = \begin{cases} 1 & |t| < 1/2 \\ (1/2 & |t| = 1/2) \\ 0 & |t| > 1/2. \end{cases}$$



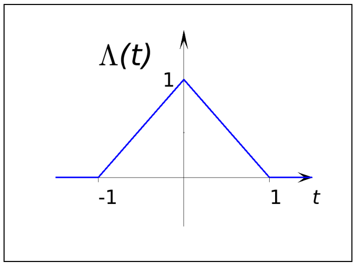
As in the case of the unit step, $\Pi(t)$ is often defined to be 1 at $|t| = 1/2$. For most applications the distinction is irrelevant. The definition given here simplifies computation of the derivative.

In multiple dimensions the **Multi-D Continuous Gate** (also known as the *Unit Square* or **Unit Cube** function) is

$$\Pi(x_1, x_2, \ldots, x_N) = \Pi(x_1)\Pi(x_2)\cdots\Pi(x_N).$$

The **Triangle** function here denoted by $\Lambda(t)$ is defined as

$$\Lambda(t) = \begin{cases} 1 - |t| & |t| < 1 \\ 0 & t \geq 1. \end{cases}$$

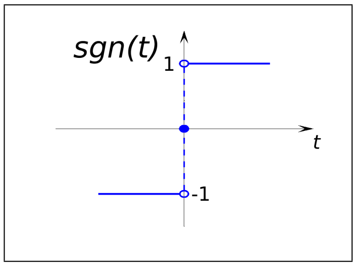


In multiple dimensions the triangle becomes the **Pyramid** function defined as

$$\Lambda(x_1, x_2, \ldots, x_N) = \Lambda(x_1)\Lambda(x_2)\cdots\Lambda(x_N).$$

The **Signum** function is denoted by sgn (t) and is defined as

$$\text{sgn}\,(t) = \begin{cases} 1 & t > 0 \\ 0 & t = 0 \\ -1 & t < 0. \end{cases}$$

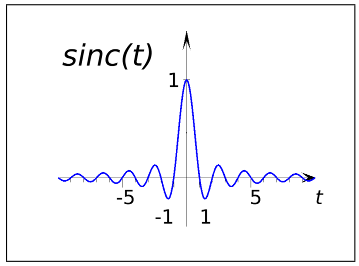


[2]Since multi-D functions are often used for spatial signals, the spatial variable x is often used as the 'time' variable instead of t.

sgn (0) is defined here consistent with Fourier analysis. However, some authors leave sgn (0) undefined.

The **Sinc** function denoted by sinc (t) is here defined as

$$\text{sinc}\,(t) = \frac{\sin \pi t}{\pi t}$$



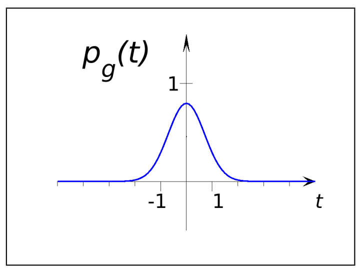
Note that some authors neglect the π in the definition of the sinc function; however, the standard definition includes π.

In multiple dimensions the **Multi-D Sinc** function is

$$\text{sinc}\,(x_1, x_2, \ldots, x_N) = \text{sinc}\,(x_1)\text{sinc}\,(x_2) \cdots \text{sinc}\,(x_N).$$

A **Unit Gaussian pulse** function denoted by $p_g(t)$ is defined as

$$p_g(t) = \frac{1}{\sqrt{2\pi}} e^{-t^2}$$

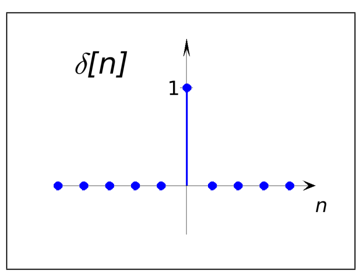


1.6.2 Discrete Functions and Sequences

Unlike the continuous case, discrete signals are always bounded. Thus, appeal to generalized functions is not required for computing the Fourier transform of discrete functions.

The **Discrete Impulse** function denoted by $\delta[n]$ is

$$\delta[n] = \begin{cases} 1 & n = 0 \\ 0 & \text{else.} \end{cases}$$



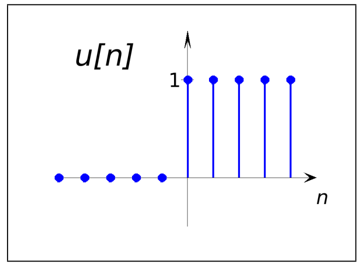
In multiple dimensions the **Multi-D Discrete Impulse** function is

$$\delta[n_1, n_2, \ldots, x_N] = \delta[n_1]\delta[n_2] \cdots \delta[n_N].$$

The **Discrete Unit Step** function, also known as the Heavyside function, is denoted by $u[n]$[3]. and is defined as

[3]The discrete unit step is also frequently denoted by $H[n]$

$$u[n] = \begin{cases} 1 & n \geq 0 \\ 0 & n < 0. \end{cases}$$

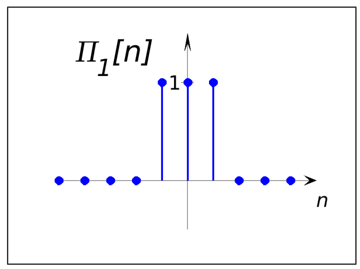


In multiple dimensions the **Multi-D Discrete Unit Step** function is

$$u[n_1, n_2, \ldots, x_N] = u[n_1]u[n_2] \cdots u[n_N].$$

The **Discrete Gate** function denoted by $\Pi_m[n]$ is

$$\Pi_m[n] = \begin{cases} 1 & |n| \leq m \\ 0 & |n| > m \end{cases}$$

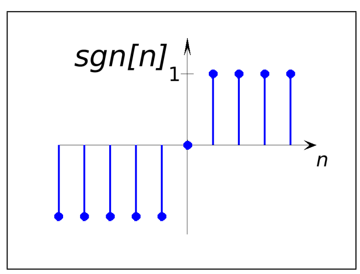


In multiple dimensions the **Multi-D Discrete Gate** function (also known as the **Discrete Square** or **Discrete Cube** is defined as,

$$\Pi_m[n_1, n_2, \ldots, n_N] = \Pi_m[n_1]\Pi_m[n_2] \cdots \Pi_m[n_N].$$

The **Discrete Signum** function denoted by $\mathrm{sgn}\,[n]$ is defined as

$$\mathrm{sgn}\,[n] = \begin{cases} 1 & n > 0 \\ 0 & n = 0 \\ -1 & n < 0. \end{cases}$$

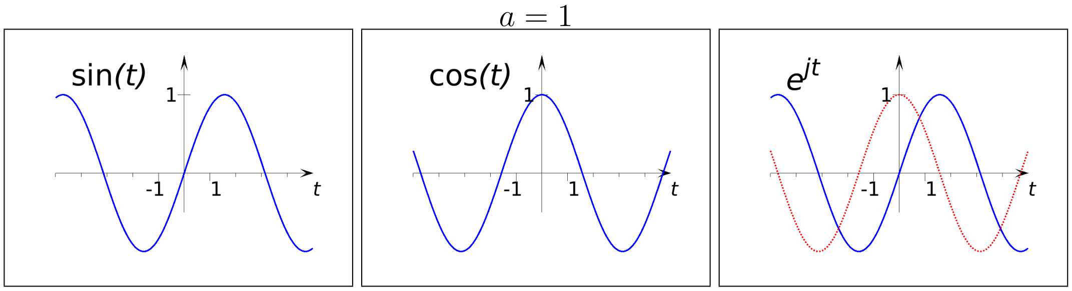


1.7 Periodic Functions

Periodic functions are functions which *exactly* repeat at evenly spaced intervals known as the *period*, i.e., if $f(t)$ is a periodic function then $f(t) = f(t + nT)\ \forall n$ where T is a multiple of the period of the function (the **period** is the smallest T for which this expression holds).

1.7.1 Standard Periodic Functions

Some standard periodic functions include the Euler functions $\cos(at)$, $\sin(at)$, and $e^{jat} = \cos(at) + j\sin(at)$ where a is a constant.



(solid blue=real part, dotted red=imaginary part)

Note: Using Euler's equations the trigonometric functions $\cos(t)$ and $\sin(t)$ may be expressed as functions of complex exponentials, i.e.,

$$\begin{aligned}\cos(t) &= \frac{1}{2}\left(e^{jt}+e^{-jt}\right)\\ \sin(t) &= \frac{j}{2}\left(e^{jt}-e^{-jt}\right).\end{aligned}$$

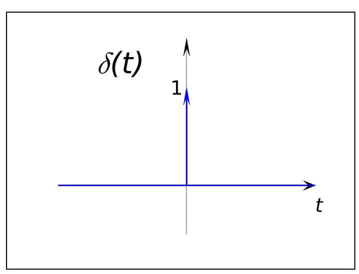
Further,

$$\begin{aligned}e^{jt} &= \cos(t)+j\sin(t)\\ e^{-jt} &= \cos(t)-j\sin(t).\end{aligned}$$

In these expressions, t is generally real. However, complex t values can be used.

1.8 Limit Functions

Oliver Heavyside, an electrical engineer/scientist of the late 19th century, first developed the concept we now call the δ function.



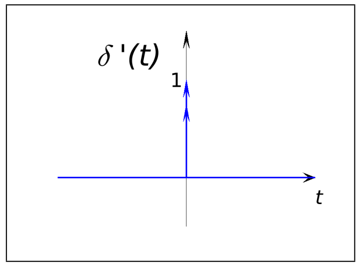
Associated with a δ function is its "weight" which is its area when integrated. When unspecified, the weight is one.

The δ function is a special case of what are somes known as *generalized functions* or *limit functions*. Such "functions" are not functions is the strict sense but are defined in terms of the limit of a sequence of function. However, they may often be usefully treated as conventional functions. The δ function is particularily useful in Fourier Analysis though its Fourier transform does not, strictly speaking, exist, though we say that it 'exists in the limit'. Additional information is given in Chapter 3.

1.8.1 Common Limit Functions

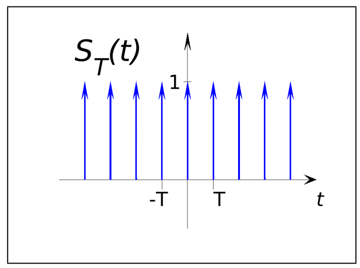
In addition to the δ function, some standard limit functions include the **Doublet** function denoted by $\delta'(t)$ is the derivative of $\delta(t)$ function, i.e.,

$$\delta'(t) = \frac{d}{dt}\delta(t)$$



and the periodic **Picket Fence** or **Sampling Function**, denoted by $S_T(t)$ is defined as,

$$S_T(t) = \sum_{k=-\infty}^{\infty} \delta(t - kT)$$



The picket fence function with $T = 1$ is sometimes known as the "shah" function and denoted as $\coprod(t)$.

Chapter 2

The Continuous Fourier Transform

2.1 Definition of the Fourier Transform

2.1.1 Single Dimension

Different authors in different fields use different mathematical definitions for the Fourier Transform[1]. In this handbook, two variations of the *forward*[2] Fourier Transform common to electrical engineering are used. The first form, known as the frequency or f form, is written in terms of the frequency variable while the second form, known as the radian or ω form, is written in terms of the radian frequency $\omega = 2\pi f$. The forward definition of the Fourier Transform $\mathcal{F}\{g(t)\} = G(f)$ [f form] or $\mathcal{F}\{g(t)\} = G(\omega)$ [ω form] of a signal $g(t)$ is defined as

$$\begin{aligned} G(f) &= \mathcal{F}\{g(t)\} = \int_{-\infty}^{\infty} g(t)e^{-j2\pi ft}dt \quad [f \text{ form}] \\ G(\omega) &= \mathcal{F}\{g(t)\} = \int_{-\infty}^{\infty} g(t)e^{-j\omega t}dt \quad [\omega \text{ form}]. \end{aligned} \tag{2.1}$$

[1]For example, a commonly used definition of the Fourier Transform pair used in Physics and Mathematics is,

$$\begin{aligned} G(\omega) &= \mathcal{F}\{g(t)\} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} g(t)e^{j\omega t}dt \\ g(t) &= \mathcal{F}^{-1}\{G(\omega)\} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} G(\omega)e^{-j\omega t}d\omega \end{aligned}$$

Note the sign of the exponentials. This form of the Fourier transform is termed the *root form* due to the square root in the leading term.

[2]Note the sign of the exponential term. In some definitions forward transform is with $e^{+j\omega t}$ rather than $e^{-j\omega t}$ as is commonly used in electrical engineering.

The corresponding *inverse* Fourier transform $g(t) = \mathcal{F}^{-1}\{G(f)\}$ [f form] or $g(t) = \mathcal{F}^{-1}\{G(\omega)\}$ [ω form] is defined as

$$\begin{aligned} g(t) &= \mathcal{F}^{-1}\{G(f)\} = \int_{-\infty}^{\infty} G(f)e^{j2\pi t}df \quad [f \text{ form}] \\ g(t) &= \mathcal{F}^{-1}\{G(\omega)\} = \frac{1}{2\pi}\int_{-\infty}^{\infty} G(\omega)e^{j\omega t}d\omega \quad [\omega \text{ form}]. \end{aligned} \tag{2.2}$$

Both the f and ω forms are frequently used. Thus, both forms are used in this book for the one-dimensional Fourier Transform. Note the signs of the arguments of the exponential functions. Conditions for the existence of the Fourier Transform are given in Chapter 3. To be short in notation, a Fourier transform pair is denoted by $g(t) \longleftrightarrow G(\omega)$.

While t is commonly "time" (in s) and f is frequency (in Hz) [$\omega = 2\pi f$ is the "radian frequency" in radians per second], a spatial variable (e.g., x) can be used for t, e.g., x can be distance in meters. Then, f is the spatial frequency in units of $1/m$. This spatial frequency is sometimes called the "wavenumber".

2.2 Existence of the Fourier Transform

Mathematical conditions for the existence of the Fourier transform can be very involved. These conditions are covered in detail by Bracewell [4] and Papoulis [19]. In this book only provides working definitions.

A sufficient, but not necessary, condition for the existence of the Fourier Transform $G(\omega)$ of $g(t)$ is that $g(t)$ is absolutely integrable, i.e.,

$$\int_{-\infty}^{\infty} |g(t)|dt < \infty,$$

and that it has only a finite number of finite discontinuities. This requirement is generally met for real-world signals. However, this requirement is not met for Euler functions or for periodic functions which, strictly speaking, do not have Fourier Transforms. Instead, the concept of the transform in the limit if often used. If the limit exists, it is the transform of the function. This convenience, often troublesome to mathematicans, is very useful in engineering applications.

Note that when the function is square integrable, i.e.,

$$\int_{-\infty}^{\infty} |g(t)|^2 dt < \infty,$$

the Fourier integral is guaranteed to be finite and that the mean squared error between the function and its Fourier representation is zero, i.e., defining $\widehat{x}(t)$ as the inverse Fourier transform of $X(\omega)$

$$\widehat{x}(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)e^{j\omega t}d\omega,$$

and the error $e(t)$ as $e(t) = x(t) - \widehat{x}(t)$, then

$$\int_{-\infty}^{\infty} |e(t)|^2 dt = 0.$$

This suggests that $\widehat{x}(t) = x(t)$ for all t except possible for a few points at which $x(t)$ is discontinuous. At these point $\widehat{x}(t)$ converges to the the average value of $x(t)$ on either side of the discontinuity.

2.2.1 Dirichlet Conditions

A more general set of requirements for convergence of the Fourier transform are known as the *Dirichlet conditions*. If these requirements are met the Fourier transform converges.

1. $x(t)$ must be absolutely integrable, i.e.,

$$\int_{-\infty}^{\infty} |x(t)| dt < \infty,$$

2. $x(t)$ can have only a finite number of maxima and minima within any finite interval.

3. $x(t)$ can have only a finite number of discontinuities in any finite interval. All discontinuities must be finite.

2.3 Multiple Dimensions

For higher-order dimensions, the ω form is most commonly used. In two dimensions, the Fourier transform pair may be expressed as

$$\begin{aligned} F(v,\omega) &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)e^{-j(vx+\omega y)}dxdy \\ f(x,y) &= \frac{1}{(2\pi)^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F(v,\omega)e^{j(vx+\omega y)}dvd\omega \end{aligned}$$

while in three dimensions the Fourier transform pair become

$$\begin{aligned} F(u,v,\omega) &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)e^{-j(ux+vy+\omega z)}dxdydz \\ f(x,y,z) &= \frac{1}{(2\pi)^3}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F(u,v,\omega)e^{j(ux+vy+\omega z)}dudvd\omega. \end{aligned}$$

In general, it is easy to see that if $\boldsymbol{x}$ and $\boldsymbol{\omega}$ are n dimensional vectors then,

$$\begin{aligned} F(\boldsymbol{\omega}) &= \int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} f(\boldsymbol{x})e^{-j\boldsymbol{x}\cdot\boldsymbol{\omega}}dx_1\cdots dx_n \\ f(\boldsymbol{x}) &= \frac{1}{(2\pi)^n}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} F(\boldsymbol{\omega})e^{j\boldsymbol{x}\cdot\boldsymbol{\omega}}d\omega_1\cdots d\omega_n. \end{aligned}$$

2.4 Key Properties of the Fourier Transform

2.4.1 One Dimension

The following table lists the even/odd properties of the Fourier transform:

$g(t)$	$\mathcal{F}\{g(t)\}$
real, even	real, even
real, odd	imag, odd
imag, even	imag, even
imag, odd	real, odd

These properties can be shown from directly from the definition of the Fourier transform.

It can easily be shown that the Fourier transform is **linear**, i.e., if $g_1(t) \longleftrightarrow G_1(f)$ and $g_2(t) \longleftrightarrow G_2(f)$, [$f$ form] then

$$\mathcal{F}\{ag_1(t) + bg_2(t)\} = aG_1(f) + bG_2(f). \tag{2.3}$$

or for the [ω form] if $g_1(t) \longleftrightarrow G_1(\omega)$ and $g_2(t) \longleftrightarrow G_2(\omega)$ then

$$\mathcal{F}\{ag_1(t) + bg_2(t)\} = aG_1(\omega) + bG_2(\omega) \tag{2.4}$$

where a and b are any constants.

Duality or **Symmetry** is also exhibited by the Fourier transform, i.e., if $g(t) \longleftrightarrow G(\omega)$ then $G(t) \longleftrightarrow 2\pi g(-\omega)$ [ω form]. To prove this, note from the definition that,

$$2\pi g(-t) = \int_{-\infty}^{\infty} G(\omega) e^{-j\omega t} d\omega \quad [\omega \text{ form}] \tag{2.5}$$

$$g(-t) = \int_{-\infty}^{\infty} G(f) e^{-j2\pi f t} df \quad [f \text{ form}] \tag{2.6}$$

Interchange the roles of t and ω (or ff) so that,

$$2\pi g(-\omega) = \int_{-\infty}^{\infty} G(t) e^{-jt\omega} dt \quad [\omega \text{ form}] \tag{2.7}$$

$$g(-f) = \int_{-\infty}^{\infty} G(t) e^{-jt\omega} dt \quad [f \text{ form}] \tag{2.8}$$

hence, $2\pi g(-\omega) \longleftrightarrow G(t)$ [ω form] and $g(-f) \longleftrightarrow G(t)$ [f form]. Note that $g(-t) \longleftrightarrow G(-\omega)$ and $g(-t) \longleftrightarrow G(-f)$.

The Fourier transform **Scaling** property yields for the real constant a,

$$g(at) \longleftrightarrow \frac{1}{|a|} G\left(\frac{\omega}{a}\right) \tag{2.9}$$

if $g(t) \longleftrightarrow G(\omega)$ [ω form] and

$$g(at) \longleftrightarrow \frac{1}{|a|} G\left(\frac{f}{a}\right) \tag{2.10}$$

if $g(t) \longleftrightarrow G(f)$ [f form].

Time Shifting can be used to simplfy the computation of the Fourier transform: if $g(t) \longleftrightarrow G(\omega)$ or then $g(t) \longleftrightarrow G(f)$

$$\begin{aligned} g(t-\tau) &\longleftrightarrow e^{-j\omega\tau}G(\omega). \quad [\omega \text{ form}] \\ g(t-\tau) &\longleftrightarrow e^{-j2\pi\tau}G(f). \quad [f \text{ form}] \end{aligned}$$

Similarly, **Frequency Shifting** yields,

$$\begin{aligned} e^{j\omega_0 t}g(t) &\longleftrightarrow G(\omega-\omega_0) \quad [\omega \text{ form}] \\ e^{j2\pi f_0 t}g(t) &\longleftrightarrow G(f-f_0) \quad [f \text{ form}]. \end{aligned}$$

These can be easily shown by substitution.

For **time differentiation** and **integration**: if $g(t) \longleftrightarrow G(\omega)$ then

$$\frac{d^n}{dt^n}g(t) \longleftrightarrow (j\omega)^n G(\omega)$$

and

$$\int_{-\infty}^{t} g(x)dx = g(t) * u(t) \longleftrightarrow \frac{G(\omega)}{j\omega} + \pi G(0)\delta(\omega).$$

For $g(t) \longleftrightarrow G(f)$ then

$$\frac{d^n}{dt^n}g(t) \longleftrightarrow (j2\pi f)^n G(f)$$

and

$$\int_{-\infty}^{t} g(x)dx = g(t) * u(t) \longleftrightarrow \frac{G(f)}{j2\pi f} + \pi G(0)\delta(f).$$

Similarly, for **frequency differentiation**:

$$(-jt)^n g(t) \longleftrightarrow \frac{d^n}{d\omega^n}G(\omega) \quad [\omega \text{ form}]$$

$$(-jt)^n g(t) \longleftrightarrow \frac{d^n}{df^n}G(f). \quad [f \text{ form}]$$

The **convolution theorem** states that if $g(t) \longleftrightarrow G(\omega)$ and $f(t) \longleftrightarrow F(\omega)$ then

$$f(t) * g(t) \longleftrightarrow F(\omega)G(\omega)$$

for the ω form and

$$f(t) * g(t) \longleftrightarrow F(f)G(f)$$

for the f form when $g(t) \longleftrightarrow G(f)$ and $f(t) \longleftrightarrow F(f)$.

Parseval's formula: if $g(t) \longleftrightarrow G(\omega)$ and $f(t) \longleftrightarrow F(\omega)$ then

$$\int_{-\infty}^{\infty} f(t)g^*(t)dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)G^*(\omega)d\omega$$

for the ω form. For the f form,

$$\int_{-\infty}^{\infty} f(t)g^*(t)dt = \int_{-\infty}^{\infty} F(f)G^*(f)df$$

when $g(t) \longleftrightarrow G(f)$ and $f(t) \longleftrightarrow F(f)$. From these formulas, the **energy theorem** provides

$$\int_{-\infty}^{\infty} |f(t)|^2 dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} |F(\omega)|^2 d\omega$$

and

$$\int_{-\infty}^{\infty} |f(t)|^2 dt = \int_{-\infty}^{\infty} |F(f)|^2 df.$$

2.4.2 Multiple Dimensions

Let $g(t_1, t_2, \ldots, t_N) \longleftrightarrow G(\omega_1, \omega_2, \ldots, \omega_N)$ and $g_2(t_1, t_2, \ldots, t_N) \longleftrightarrow G_2(\omega)$.

The multi-dimensional Fourier transform is **linear**, i.e.,

$$\mathcal{F}\{ag(t_1, t_2, \ldots, t_N) + bg_2(t_1, t_2, \ldots, t_N)\} = aG(\omega_1, \omega_2, \ldots, \omega_N) + bG_2(\omega_1, \omega_2, \ldots, \omega_N)$$

where a and b are any constants (which may be complex).

Duality or **Symmetry**

$$G(t_1, t_2, \ldots, t_N) \longleftrightarrow (1/2\pi)^N g(-\omega_1, -\omega_2, \ldots, -\omega_N).$$

The **Scaling** property yields for the real constants a_i,

$$g(a_1 t_1, a_2 t_2, \ldots, a_N t_N) \longleftrightarrow \frac{1}{|a_1|}\frac{1}{|a_2|}\cdots\frac{1}{|a_N|} G\left(\frac{\omega_1}{a_1}, \frac{\omega_2}{a_2}, \ldots, \frac{\omega_N}{a_N}\right).$$

Time Shifting can be used to simplfy the computation of the Fourier transform for real τ_i:

$$g(t_1 - \tau_1, t_2 - \tau_2, \ldots, t_N - \tau_N) \longleftrightarrow e^{-j(\omega_1\tau_1 + \omega_2\tau_2 + \cdots + \omega_N\tau_N)} G(\omega_1, \omega_2, \ldots, \omega_N).$$

Similarly, **Frequency Shifting** yields,

$$e^{j(\tau_1 t_1 + \tau_2 t_2 + \cdots + \tau_N t_N)} g(t_1, t_2, \ldots, t_N) \longleftrightarrow G(\omega_1 - \tau_1, \omega_2 - \tau_2, \ldots, \omega_N - \tau_N).$$

For **time differentiation** and **integration**:

$$\frac{\partial}{\partial t^{n_1}}\frac{\partial}{\partial t^{n_2}}\cdots\frac{\partial}{\partial t^{N_1}} g(t_1, t_2, \ldots, t_N) \longleftrightarrow (j\omega_1)^{n_1}(j\omega_2)^{n_2}\cdots(j\omega_N)^{n_N} G(\omega_1, \omega_2, \ldots, \omega_N)$$

and

$$\int_{-\infty}^{t} g(x)dx = g(t) * u(t) \longleftrightarrow \frac{G(\omega)}{j\omega} + \pi G(0)\delta(\omega).$$

Similarly, for **frequency differentiation**:

$$(-jt_1)^{n_1}(-jt_2)^{n_2}\cdots(-jt_N)^{n_N} g(t_1, t_2, \ldots, t_N) \longleftrightarrow \frac{\partial}{\partial \omega^{n_1}}\frac{\partial}{\partial \omega^{n_2}}\cdots\frac{\partial}{\partial \omega^{N_1}} G(\omega_1, \omega_2, \ldots, \omega_N).$$

Multi-dimensional convolution $f * g$ of $f(t_1, t_2, \ldots, t_N)$ and $g(t_1, t_2, \ldots, t_N)$ is defined as,

$$f * g = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} f(t_1 - \tau_1, t_2 - \tau_2, \ldots, t_N - \tau_N) g(\tau_1, \tau_2, \ldots, \tau_N) d\tau_1 d\tau_2 \cdots d\tau_N.$$

In multiple dimensions, the **convolution theorem** states that if $f(t_1, t_2, \ldots, t_N) \longleftrightarrow F(\omega_1, \omega_2, \ldots, \omega_N)$ and $g(t_1, t_2, \ldots, t_N) \longleftrightarrow G(\omega_1, \omega_2, \ldots, \omega_N)$ then

$$f * g \longleftrightarrow F(\omega_1, \omega_2, \ldots, \omega_N) G(\omega_1, \omega_2, \ldots, \omega_N).$$

Parseval's formula: if $f(t_1, t_2, \ldots, t_N) \longleftrightarrow F(\omega_1, \omega_2, \ldots, \omega_N)$ and $g(t_1, t_2, \ldots, t_N) \longleftrightarrow G(\omega_1, \omega_2, \ldots, \omega_N)$ then

$$\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} f(t_1, t_2, \ldots, t_N) g^*(t_1, t_2, \ldots, t_N) dt_1 dt_2 \cdots dt_N =$$

$$\frac{1}{(2\pi)^N} \int_{-\infty}^{\infty} \cdots\cdots \int_{-\infty}^{\infty} F(\omega_1, \omega_2, \ldots, \omega_N) G^*(\omega_1, \omega_2, \ldots, \omega_N) d\omega_1 d\omega_2 \cdots d\omega_N.$$

2.5 Computation of the Fourier Transform

Computing the Fourier transform of a given $g(t)$ or determining the inverse transform of $G(\omega)$ can require skill. There are a wide variety of techniques and "tricks" which can be used to simplify the problem. The primary techniques are 1) the direct method, 2) using the "sifting" property of the δ function, or 3) using look up tables. Of these three, the latter is usually chosen. Examples of the application of each of these techniques is given in the following sections.

2.5.1 The Direct Method

The *direct method*, also known as the *integral method*, consists of directly applying the equations defining the Fourier transform or it's inverse. As an example, using the ω form, let $g(t) = p_\tau(t)$ where

$$p_\tau(t) = \begin{cases} 1 & -\tau \le t \le \tau \\ 0 & \text{else.} \end{cases}$$

$G(\omega) = \mathcal{F}\{g(t)\}$ is then,

$$\begin{aligned} G(\omega) &= \int_{-\tau}^{\tau} e^{-j\omega t} dt = -\frac{1}{j\omega} e^{-j\omega t}\Big|_{t=-\tau}^{\tau} \\ &= \frac{1}{j\omega}\left(e^{j\omega\tau} - e^{-j\omega\tau}\right) = \frac{2}{\omega}\sin(\omega\tau) \\ &= 2\tau \frac{\sin(\omega\tau)}{\omega\tau} = 2\tau \operatorname{sinc}(\frac{\omega\tau}{\pi}). \end{aligned}$$

2.5.2 Sifting Property of the δ Function

The *sifting property* of the δ function (see Chapter 4) is that

$$\int_{-\infty}^{\infty} \delta(t-a)f(t)dt = f(a).$$

Using this property the Fourier transform of $\delta(t)$ is

$$\begin{aligned}\mathcal{F}\{\delta(t)\} &= \int_{-\infty}^{\infty} \delta(t)e^{j\omega t}dt \\ &= e^{j\omega t}|_{t=0} = 1\end{aligned}$$

and related signals such as $S_\delta(t) = \coprod(t)$,

$$\begin{aligned}\mathcal{F}\{S_\delta(t)\} &= \int_{-\infty}^{\infty} S_\delta(t)e^{j\omega t}dt \\ &= \sum_{n=-\infty}^{\infty} e^{j\omega nT} \\ &= \frac{2\pi}{T}\sum_{n=-\infty}^{\infty} \delta(\omega - n\omega_0) \\ &= \frac{2\pi}{T}\coprod(t).\end{aligned}$$

2.5.3 The Table-Based Method

The table-based method of computing the Fourier Transform, sometimes known as the *transform method*, might be called the “divide and conquer” method. Computing the Fourier transform of general signals can be simplified by using a table of previously computed transformations. This method is based on the linearity of the Fourier transform and it’s inverse. Using a table of known transforms, a composite function is constructed of known functions using linear operations. The tranform is then computed by applying the same operations to the coresponding transforms of the individual function components.

As a simple example, note that the inverse Fourier transform of $\delta(\omega)$ is $1/2\pi$; hence, $1 \longleftrightarrow 2\pi\delta(\omega)$.

Remember that finite length signals ALWAYS *have infinite length Fourier transforms. Finite length Fourier transforms* ALWAYS *have infinite length inverse transforms.* However, converse is not true: infinite length signals can have infinite length Fourier transforms.

2.6 Alternate Forms of the Fourier Transform

The Fourier Transform may be applied to a general class of functions. For special classes of functions, other Fourier analysis techniques have been developed to simplify analysis and provide additional insight. These include the *Fourier Series* (FS), the *Discrete Fourier Series* (DFS), *Discrete Fourier Transform* (DFT), and the *Discrete Time Discrete Fourier*

Transform (DTFT). The FS is applicable for continuous, periodic signals. The DTFT, which is applicable for general discrete signals, is the Fourier transform for discrete signals. The DFS is applicable to discrete periodic signals and is similar to the Fourier series. The DFT is conceptually similar to the DFS. It assumes a discrete, periodic signal. When applied to an aperiodic signal, the frequency domain representation corresponds to a periodic extention of the input signal. The *Fast Fourier Transform* (FFT) is an efficient numerical implementation of the DFT. The Fourier Series is developed in Chapter 3 and Chapter 5 while the DFS, DFT, DTFT, and the FFT are considered in Chapter 4. Generalizations of the Fourier transform, the Laplace and $\mathcal{Z}$ transforms, are considered in Chapter 6.

The Table 2.1 summarizes the applicability of each form of the Fourier Transform. Note that by using the limit concept the Fourier transform of a Fourier series representation of a periodic signal can be computed. Similarly, the discrete-time Fourier transform can be computed from the discrete Fourier series representation of a discrete, periodic signal.

Table 2.1: Signal Class and Applicable Fourier Transform

Signal class	**Applicable transform**
Continuous	Fourier Transform
Continuous, Periodic	Fourier Series $\longrightarrow$ Fourier Transform
Discrete	Discrete Time Fourier Transform
Discrete, Periodic	Discrete Time Fourier Series $\longrightarrow$ Discrete Time Fourier Transform
	Discrete Fourier Transform

The frequency domain representation is somewhat different for each transform. Table 2.2 summarizes the frequency domain representation for each transform. While the Fourier transform's frequency domain is, in general, continuous and aperiodic, when applied to periodic or discrete signals using limit techniques, the frequency domain representation may become periodic and/or discrete. Table 2.3 further summarizes the relationship between the time domain signal properties, the appropriate Fourier transform, and the frequency domain properties.

Table 2.2: Fourier Transform and Frequency Domain Properties

Transform	**Frequency Domain Properties**
Fourier Transform	Continuous
Fourier Series	Discrete, Periodic
Discrete Time Fourier Transform	Periodic
Discrete Time Fourier Series	Periodic, Discrete
Discrete Fourier Transform	Periodic, Discrete

Table 2.3: Relationship Between Time Domain Signal Properties and It's Frequency Domain Properties for Each Fourier Transform

Time Domain	**Transform**	**Frequency Domain**
Continuous, Aperiodic	Fourier Transform	Aperiodic, Continuous
Continuous, Periodic	Fourier Series	Aperiodic, Discrete
	Fourier Transform	Aperiodic, Discrete*
Discrete, Aperiodic	Discrete Time Fourier Transform	Periodic, Continuous
Discrete, Periodic	Discrete Time Fourier Series	Periodic, Discrete
	Discrete Time Fourier Transform	Periodic, Discrete*
Discrete, Periodic†	Discrete Time Fourier Transform	Periodic, Discrete

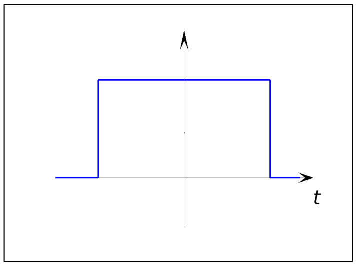
* δ functions
† assumed periodic

2.7 Fourier Transform Tables

The following sections provide tables of Fourier transform pairs.

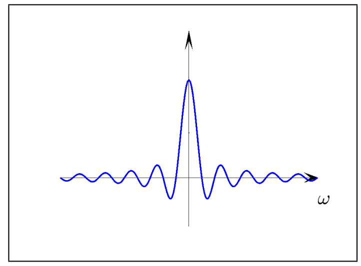
2.7.1 Pictitorial Fourier Transforms

These pictitorial exampes are for the ω form. Real parts are shown in solid blue while imaginary parts (if any) are shown in dotted red.

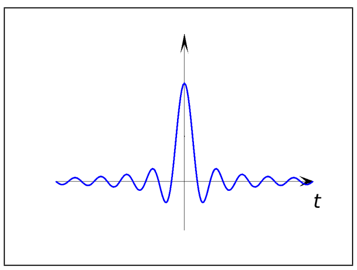


$u(t)$

$\longleftrightarrow$

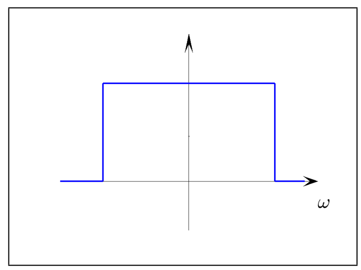


$2\text{sinc}(\omega/2)$

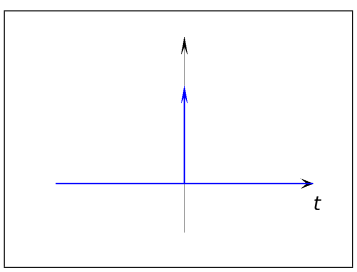


$\text{sinc}\,(t)$

$\longleftrightarrow$

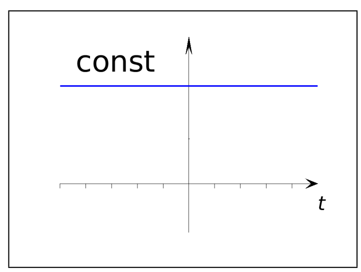


$\Pi(\omega/2\pi)$

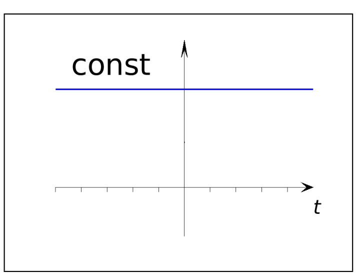


$\delta(t)$

$\longleftrightarrow$

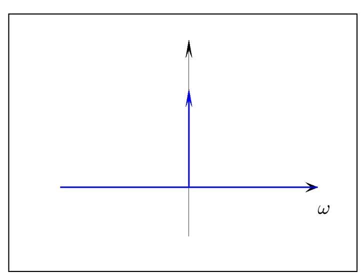


1 (constant)

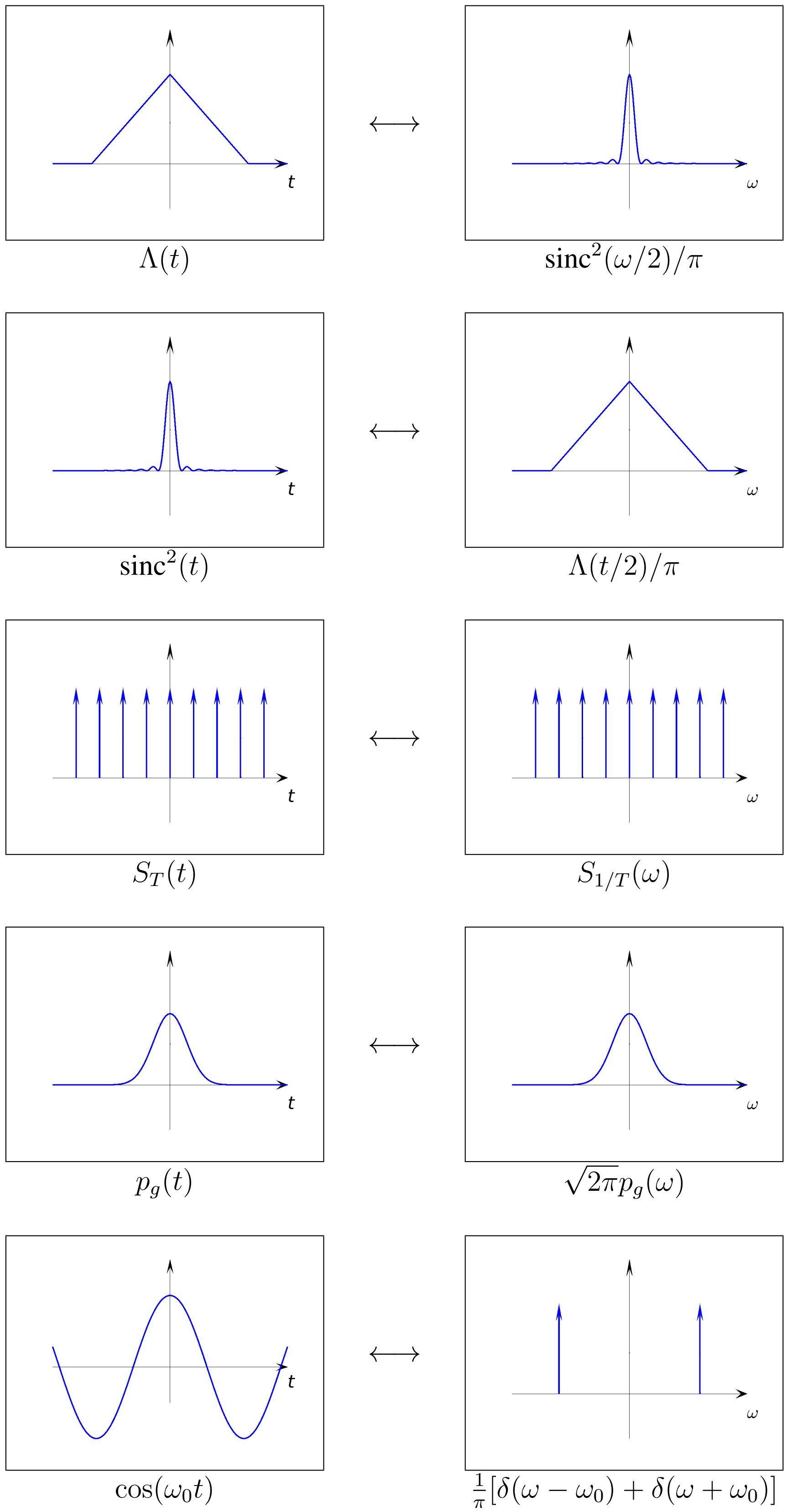


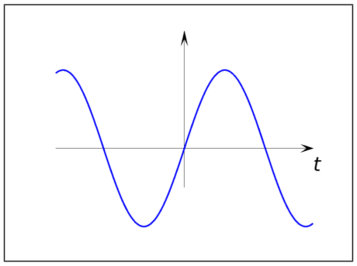
1 (constant)

$\longleftrightarrow$



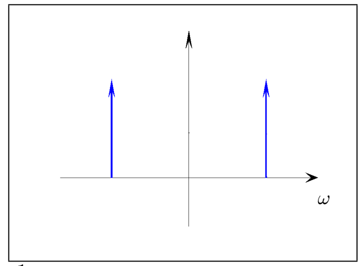
$\delta(\omega)$


$\Lambda(t)$ $\longleftrightarrow$ $\text{sinc}^2(\omega/2)/\pi$
$\text{sinc}^2(t)$ $\longleftrightarrow$ $\Lambda(t/2)/\pi$
$S_T(t)$ $\longleftrightarrow$ $S_{1/T}(\omega)$
$p_g(t)$ $\longleftrightarrow$ $\sqrt{2\pi}p_g(\omega)$
$\cos(\omega_0 t)$ $\longleftrightarrow$ $\frac{1}{\pi}[\delta(\omega-\omega_0)+\delta(\omega+\omega_0)]$

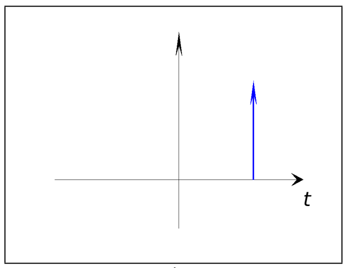


$\sin(\omega_0 t)$

$\longleftrightarrow$

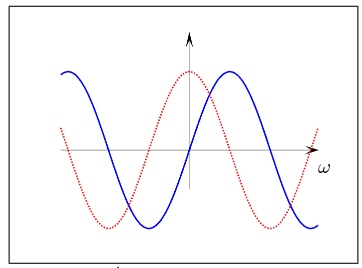


$j\frac{1}{\pi}\pi[\delta(\omega-\omega_0)-\delta(\omega+\omega_0)]$

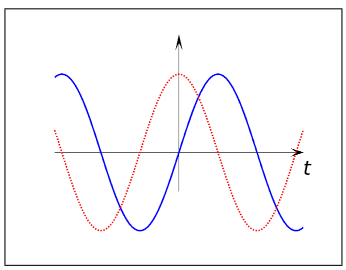


$e^{-j\omega_0 t}$

$\longleftrightarrow$

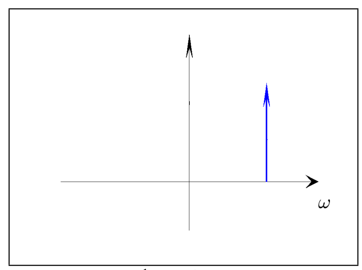


$\frac{1}{\pi}\delta(\omega-\omega_0)$



$\delta(\omega-\omega_0)$

$\longleftrightarrow$



$\frac{1}{\pi}e^{-j\omega_0 t}$

2.7.2 One-Dimensional Fourier Transform (f Form)

Definition and Key Properties

	$g(t) = \mathcal{F}^{-1}\{G(f)\}$	$G(f) = \mathcal{F}\{g(t)\}$
1	$g(t)$	$\int_{-\infty}^{\infty} g(t)e^{-j2\pi ft}dt$
2	$\int_{-\infty}^{\infty} G(f)e^{j2\pi ft}df$	$G(f)$
3	$G(t)$	$g(-f)$
4	$G(-t)$	$g(f)$
5	$g(at)$	$\frac{1}{\lvert a\rvert}G(\frac{f}{a})$
6	$g(t-t_o)$	$e^{-j2\pi ft_o}G(f)$
7	$g(t)e^{j2\pi f_o t}$	$G(f-f_o)$
8	$g(t)e^{j(2\pi f_o t+\phi)}$	$G(f-f_o)e^{j\phi}$
9	$\frac{d^n}{dt^n}g(t)$	$(j2\pi f)^n G(f)$
10	$g_1(t) * g_2(t)$	$G_1(f)G_2(f)$
11	$g_1(t)g_2(t)$	$G_1(f) * G_2(f)$
12	$g(-t)$	$G(-f) = G^*(f)$
13	$\int_{-\infty}^{t} g(\tau)d\tau$	$\frac{1}{j2\pi f}G(f) + \frac{1}{2}G(0)\delta(f)$

1-D Fourier Transform Table (f Form)

1-D Fourier Transform Table (f Form)

	$g(t) = \mathcal{F}^{-1}\{G(f)\}$	$G(f) = \mathcal{F}\{g(t)\}$
1	$e^{-at}u(t)$	$\frac{1}{(a + j2\pi f)}$
4	$\delta(t)$	1
5	1	$\delta(f)$
6	$u(t)$	$\frac{1}{2}\delta(f) + \frac{1}{j2\pi f}$
7	$\cos 2\pi f_o t$	$\frac{1}{2}[\delta(f - f_o) + \delta(f + f_o)]$
8	$\sin 2\pi f_o t$	$\frac{1}{2j}[\delta(f - f_o) - \delta(f + f_o)]$
12	$2B\text{sinc}(2Bt)$	$\Pi\left(\frac{f}{2B}\right)$
13	$\Pi\left(\frac{t}{\tau}\right) = \begin{cases} 1 & \lvert t\rvert < \tau/2 \\ 0 & \text{else} \end{cases}$	$\tau\text{sinc}(f\tau)$
14	$\Lambda\left(\frac{t}{\tau}\right) = \begin{cases} 1 - \lvert t\rvert/\tau & \lvert t\rvert < \tau \\ 0 & \text{else} \end{cases}$	$\tau\text{sinc}^2(f\tau)$
15	$\text{sinc}^2(f_o t)$	$\frac{1}{f_o}\Lambda\left(\frac{f}{f_o}\right)$
16	$e^{-a\lvert t\rvert}$	$\frac{2a}{a^2 + (2\pi f)^2}$
17	e^{-t^2/σ^2}	$\sigma e^{-\pi\sigma^2 f^2}$
18	$\text{sgn}(t)$	$\frac{1}{j\pi f}$
19	$e^{j(2\pi f_o t + \phi)}$	$\delta(f - f_o)e^{j\phi}$
20	$\sum_{k=-\infty}^{\infty} \delta(t - kT)$	$\frac{1}{T}\sum_{n=-\infty}^{\infty} \delta(f - n\frac{1}{T})$
21	$\frac{1}{t}$	$-\frac{j}{2\text{sgn}(f)}$
23	$\frac{d}{dt}f(t) * \frac{1}{\pi t}$	$\lvert f\rvert F(f)$
24	$\alpha/(\alpha^2 + t^2)$	$\pi e^{-\alpha\lvert f\rvert}$
25	$t^n e^{-\alpha t}u(t)$	$\frac{n!}{(\alpha + jf)^{n+1}}$
26	$\frac{-1}{\pi t^2}$	$\lvert w\rvert$
29	$te^{-j\alpha t}$	$(\alpha + jf)^{-2}$
30	$tu(t)$	$-f^{-2} + \pi j\delta'(f)$

2.7.3 One-Dimensional Fourier Transform (ω Form)

Definition and Key Properties

	$g(t) = \mathcal{F}^{-1}\{G(\omega)\}$	$G(\omega) = \mathcal{F}\{g(t)\}$
1	$f(t)$	$\int_{-\infty}^{\infty} g(t)e^{-j\omega t}dt$
2	$\frac{1}{2\pi}\int_{-\infty}^{\infty} G(\omega)e^{j\omega t}d\omega$	$G(\omega)$
3	$G(t)$	$2\pi g(-\omega)$
4	$\frac{1}{2\pi}G(-t)$	$g(\omega)$
5	$g(at)$	$\frac{1}{\lvert a\rvert}G(\omega)$
6	$g(t-t_o)$	$e^{-j\omega t_o}G(\omega)$
7	$g(t)e^{-j\omega_o t}$	$G(\omega-\omega_o)$
8	$\frac{d^n}{dt^n}g(t)$	$(j\omega)^n G(\omega)$
9	$g_1(t) * g_2(t)$	$G_1(\omega)G_2(\omega)$
10	$g_1(t)g_2(t)$	$\frac{1}{2\pi}G_1(\omega) * G_2(\omega)$

1-D Fourier Transform Table (ω Form)

1-D Fourier Transform Table (ω Form)

	$g(t) = \mathcal{F}^{-1}\{G(\omega)\}$	$G(\omega) = \mathcal{F}\{g(t)\}$
1	$e^{-at}u(t)$	$\dfrac{1}{(a+j\omega)}$
2	$te^{-at}u(t)$	$1/(a+j\omega)^2$
3	$\|t\|$	$-2\omega^{-2}$
4	$\delta(t)$	1
5	1	$2\pi\delta(\omega)$
6	$u(t)$	$\pi\delta(\omega) + \dfrac{1}{j\omega}$
7	$\cos\omega_o t$	$\pi[\delta(\omega-\omega_o) + \delta(\omega+\omega_o)]$
8	$\sin\omega_o t$	$j\pi[\delta(\omega-\omega_o) - \delta(\omega+\omega_o)]$
9	$\cos\omega_o t u(t)$	$\dfrac{\pi}{2}[\delta(\omega-\omega_o) + \delta(\omega+\omega_o)] + j\omega/(\omega_o^2 - \omega^2)$
10	$\sin\omega_o t u(t)$	$j\dfrac{\pi}{2}[\delta(\omega-\omega_o) + \delta(\omega+\omega_o)] + \omega/(\omega_o^2 - \omega^2)$
11	$e^{-at}\sin\omega_o t u(t)$	$\dfrac{\omega_o}{(a+j\omega)^2 + \omega_0^2}$
12	$2B\text{sinc}(2Bt)$	$\Pi\left(\dfrac{\omega}{4\pi B}\right)$
13	$\Pi\left(\dfrac{t}{\tau}\right) = \begin{cases} 1 & \|t\| < \tau/2 \\ 0 & \text{else} \end{cases}$	$\tau\text{sinc}\left(\dfrac{\omega\tau}{2\pi}\right)$
14	$\Lambda\left(\frac{t}{\tau}\right) = \begin{cases} 1 - \|t\|/\tau & \|t\| < \tau \\ 0 & \text{else} \end{cases}$	$\tau\text{sinc}^2\left(\dfrac{\omega\tau}{2\pi}\right)$
15	$\text{sinc}^2\left(\dfrac{\omega_o t}{2\pi}\right)$	$\dfrac{2\pi}{\omega_o}\Lambda\left(\dfrac{\omega}{\omega_o}\right)$
16	$e^{-a\|t\|}$	$\dfrac{2a}{a^2+\omega^2}$
17	$e^{-t^2/2\sigma^2}$	$\sigma\sqrt{2\pi}e^{-\sigma^2\omega^2/2}$
18	$\text{sgn}(t)$	$\dfrac{2}{j\omega}$
19	$e^{j\omega_o t}$	$2\pi\delta(\omega-\omega_o)$
20	$\displaystyle\sum_{k=-\infty}^{\infty}\delta(t-kT)$	$\dfrac{2\pi}{T}\displaystyle\sum_{n=-\infty}^{\infty}\delta\left(\omega - n\dfrac{2\pi}{T}\right)$
21	$\dfrac{1}{t}$	$-\pi j\text{sgn}(\omega)$

1-D Fourier Transform Table (ω Form) [Continued]

	$g(t) = \mathcal{F}^{-1}\{G(\omega)\}$	$G(\omega) = \mathcal{F}\{g(t)\}$
22	$\frac{j}{\pi t}$	$\mathrm{sgn}(\omega)$
23	$\frac{d}{dt} f(t) * \frac{1}{\pi t}$	$\lvert\omega\rvert F(\omega)$
24	$\alpha/(\alpha^2 + t^2)$	$\pi e^{-\alpha\lvert\omega\rvert}$
25	$t^n e^{-\alpha t} u(t)$	$\frac{n!}{(\alpha + j\omega)^{n+1}}$
26	$\frac{-1}{\pi t^2}$	$\lvert w\rvert$
27	$\frac{1}{t}$	$-j\pi \,\mathrm{sgn}\, \omega$
28	$\lvert t\rvert e^{-\alpha\lvert t\rvert}$	$\frac{2(\alpha^2 - \omega^2)}{\alpha^2 + \omega^2}$
29	$te^{-j\alpha t}$	$(\alpha + j\omega)^{-2}$
30	$tu(t)$	$-\omega^{-2} + \pi j \delta'(\omega)$
31	$\frac{1}{\sqrt{2\pi}\sigma} e^{-t^2/(2\sigma^2)}$	$e^{-(\omega\sigma)^2/2}$
32	$p_g(t)$	$\sqrt{2\pi} p_g(\omega)$
33	$\frac{1}{t^2}$	$-\pi\lvert w\rvert$
34	$p_a(t)\left[1 + \cos(\frac{\pi}{a} t)\right]$	$\frac{2\pi^2 \sin a\omega}{\omega(\pi^2 - a\omega^2)}$

2.7.4 One-Dimensional Fourier Transform (Root Form)

Definition and Key Properties

	$g(t) = \mathcal{F}^{-1}\{G(\omega)\}$	$G(\omega) = \mathcal{F}\{g(t)\}$
1	$f(t)$	$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} g(t)e^{j\omega t}dt$
2	$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} G(\omega)e^{-j\omega t}d\omega$	$G(\omega)$
3	$G(t)$	$g(-\omega)$
4	$G(-t)$	$g(\omega)$
5	$g(at)$	$\frac{1}{\lvert a\rvert}G(\omega)$
6	$g(t-t_o)$	$e^{-j\omega t_o}G(\omega)$
7	$g(t)e^{-j\omega_o t}$	$G(\omega-\omega_o)$
8	$\frac{d^n}{dt^n}g(t)$	$(j\omega)^n G(\omega)$
9	$g_1(t) * g_2(t)$	$G_1(\omega)G_2(\omega)$
10	$g_1(t)g_2(t)$	$G_1(\omega) * G_2(\omega)$

2.7.5 Two-Dimensional Fourier Transform

2.7.6 Definition and Key Properties

	$g(t_1,t_2)=\mathcal{F}^{-1}\{G(\omega_1,\omega_2)\}$	$G(\omega_1,\omega_2)=\mathcal{F}\{g(t_1,t_2)\}$
1	$g(t_1,t_2)$	$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(t_1,t_2)e^{-j(\omega_1 t_1+\omega_2 t_2}dt_1 dt_2$
2	$\frac{1}{(2\pi)^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} G(\omega_1,\omega_2)e^{j(\omega_1 t_1+\omega_2 t_2)}d\omega_1 d\omega_2$	$G(\omega_1,\omega_2)$
3	$G(t1,t2)$	$(2\pi)^2 g(-\omega_1,-\omega_2)$
4	$\frac{1}{(2\pi)^2}G(-t_1,-t_2)$	$g(\omega_1,\omega_2)$
5	$g(at_1,bt_2)$	$\frac{1}{\lvert ab\rvert}G(\omega_1,\omega_2)$
6	$g(t_1-\tau_1,t_2-\tau_2)$	$e^{-j(\omega_1\tau_1+\omega_2\tau_2)}G(\omega_1,\omega_2)$
7	$g(t_1,t_2)e^{-j(v_1 t_1+v_2 t_2)}$	$G(\omega_1-v_1,\omega_2-v_2)$
8	$\frac{d^m}{dt_1^m}\frac{d^n}{dt_2^n}g(t_1,t_2)$	$(j\omega_1)^m(j\omega_2)^n G(\omega_1,\omega_2)$
9	$g_1(t_1,t_2)*g_2(t_1,t_2)$	$G_1(\omega_1,\omega_2)G_2(\omega_1,\omega_2)$
10	$g_1(t_1,t_2)g_2(t_1,t_2)$	$\frac{1}{(2\pi)^2}G_1(\omega_1,\omega_2)*G_2(\omega_1,\omega_2)$
11	$g(t_1,t_2)*g(-t_1,-t_2)$	$\lvert G(\omega_1,\omega_2)\rvert^2$
12	$g_1(t_1)g_2(t_2)$	$G_1(\omega_1)G_2(\omega_2)$

2-D Fourier Transform Table

2-D Fourier Transform Table

	$g(t_1,t_2)=\mathcal{F}^{-1}\{G(\omega_1,\omega_2)\}$	$G(\omega_1,\omega_2)=\mathcal{F}\{g(t_1,t_2)\}$
1	$\delta(t_1,t_2)$	1
2	$\operatorname{sinc} t_1 \operatorname{sinc} t_2$	$\Pi(\omega_1,\omega_2)$
3	$\operatorname{sinc}^2 t_1 \operatorname{sinc}^2 t_2$	$\Lambda(\omega_1,\omega_2)$
4	$\cos 2\pi t_1 \sin 2\pi t_2$	$\frac{1}{2}[\delta(\omega_1-1/2,\omega_2-1/2)+\delta(\omega_1-1/2,\omega_2+1/2)+$ $\delta(\omega_1+1/2,\omega_2-1/2)+\delta(\omega_1+1/2,\omega_2+1/2)]$
5	$\operatorname{sinc}^2 t_1 \operatorname{sinc} t_2$	$\Lambda(\omega_1)\Pi(\omega_2)$
6	$\delta(t_1)$	$\delta(\omega_2)$
7	$e^{-\pi\left(a^2\omega_1^2+b^2\omega_2^2\right)}$	$abe^{-\pi\left(\frac{t_1^2}{a^2}+\frac{t_2^2}{b^2}\right)}$

Note that additional transforms may be generated by interchanging t_1 and t_2 and ω_1 and ω_2, respectively.

2.7.7 Multi-Dimensional Fourier Transform

Definition and Key Properties

	$g(t_1,\ldots,t_N)=\mathcal{F}^{-1}\{G(\omega_1,\ldots,\omega_N)\}$	$G(\omega_1,\ldots,\omega_N)=\mathcal{F}\{g(t_1,\ldots,t_N)\}$
1	$g(t_1,\ldots,t_N)$	$\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} g(t_1,\ldots,t_N)$ $e^{-j(\omega_1 t_1+\omega_2 t_2} dt_1\cdots dt_N$
2	$\frac{1}{(2\pi)^N}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} G(\omega_1,\ldots,\omega_N)$ $e^{j(\omega_1 t_1+\cdots+\omega_N t_N)} d\omega_1\cdots d\omega_N$	$G(\omega_1,\ldots,\omega_N)$
3	$g_1(t_1)\cdots g_N(t_N)$	$G_1(\omega_1)\cdots G_N(\omega_N)$

Other properties are easily seen from the one- and two-dimensional cases.

n-D Fourier Transform Table

n-D Fourier Transform Table

	$g(t_1,\ldots,t_N)=\mathcal{F}^{-1}\{G(\omega_1,\ldots,\omega_N)\}$	$G(\omega_1,\ldots,\omega_N)=\mathcal{F}\{g(t_1,\ldots,t_N)\}$	Notes
1	$\delta(t_1,\ldots,t_N)$	1	point
2	$\text{sinc}\, t_1 \cdots \text{sinc}\, t_N$	$\Pi(\omega_1,\ldots,\omega_N)$	cube
3	$\text{sinc}^2\, t_1 \cdots \text{sinc}^2\, t_N$	$\Lambda(\omega_1,\ldots,\omega_N)$	prism
4	$\delta(t_1-\tau_1,\ldots,t_N-\tau_N)$	$e^{j2\pi(\tau_1\omega_1+\cdots+\tau_N\omega_N)}$	offset point
5	$e^{-\pi(\omega_1^2/a_1^2+\cdots+\omega_N^2/a_N^2)}$	$a_1\cdots a_N e^{-\pi(\omega_1^2/a_1^2+\cdots+\omega_N^2/a_N^2)}$	Gaussian
6	$\Pi(t/2)$	$\frac{1}{2\pi^2\omega^3}\left(\sin 2\pi\omega - 2\pi\omega\cos 2\pi\omega\right)$	ball
7	$e^{-\pi t^2}$	$e^{-\pi\omega^2}$	
8	$\Pi(t_1,\ldots,t_N)$	$\text{sinc}\,\omega_1 \cdots \text{sinc}\,\omega_N$	cube
9	$\Pi(t_1,\ldots,t_{N-1})$	$\text{sinc}\,\omega_1 \cdots \text{sinc}\,\omega_{N-1}\delta(\omega_N)$	bar
10	$Pi(t_1)$	$\text{sinc}\,\omega_1\, \delta(t_2,\ldots,t_N)$	slab

Where $t^2=t_1^2+\cdots+t_n^2$ and $\omega^2=\omega_1^2+\cdots+\omega_N^2$. Additional transforms may be generated by permutation of the t_i's and corresponding ω_i's.

Chapter 3

The Fourier Transform as a Limit

In general, the sufficient condition described above may be violated because (1) the function has infinite energy (e.g., a periodic function), or (2) because it contains δ or impulse functions or other generalized limit functions. Both exceptions, however, are usefull in engineering applications. Viewing the Fourier transform as a limit process enables simplified application Fourier analysis in these cases that they can be treated more like conventional cases, i.e.,

$$F(\omega) = \lim_{T \longrightarrow \infty} \int_{-T}^{T} f(t) e^{-j\omega t} dt.$$

Using this definition the "transforms" of periodic functions and generalized functions can be determined. In the following subsections, generalized functions are first considered followed by periodic functions. Later sections consider how the Fourier transform can be considered as the limiting case of the Fourier Series and Gibb's phenomenon.

3.1 Integral Limits and Generalized Functions

The δ or impulse "function" is not, strictly speaking, a function at all, but is a generalized function. A generalized function is defined in terms of the class of all equivalent regular sequences of particularily well-behaved functions. A particularily well-behaved function $f(t)$ is bounded by $|t|^N$ as for any large N as $t \longrightarrow \infty$. A regular sequence $f_x(t)$ of well-behaved functions have the property that

$$\lim_{x \longrightarrow 0} \int_{-\infty}^{\infty} f_x(t) g(t) dt$$

exists for any well-behaved function $g(t)$. Strictly speaking, this definition precludes simple sequences such as $\Pi(t/\tau)$ as τ goes to 0 (which is often used to develop the δ function) since $\Pi(t/\tau)$ is not well-behaved. However, so long as the derivative is not required, this approach is useful. For further information see Bracewell [4].

It is important to note that a generalized function is *not* defined in terms of the limit of a single sequence of functions but rather as a *class* of equivalent functions. For example, the δ function may be (loosely) defined in terms of the limit of $\Pi(t/\tau)$ as τ goes to 0 *or* *equivalently* in terms of the limit of $\tau^{-1}e^{\pi t^2/\tau^2}$ as τ goes to 0. The are an infinite number of equivalent approaches to defining the δ function.

Note that whenever a generalized function $f(t)$ appears in an integral, the integral should be interpreted as the limit of the integral, i.e., the integral

$$\int f(t)g(t)dt$$

should be interpreted as

$$\lim_{x \longrightarrow 0} \int f_x(t)g(t)dt$$

where $f_x(t)$ is the sequence defining $f(t)$. Hence, the Fourier transform is treated a limit for the case of a generalized function.

A particular property of the generalized function $\delta(t)$ is the so-called "sifting" or "sampling" property, i.e.,

$$\int_{-\infty}^{\infty} \delta(t-\tau)f(t)dt = f(\tau).$$

Note that whenever a generalized function appears in an integral the integral should be intepreted as a limit; hence, this equation should be understood to be,

$$\lim_{x \longrightarrow 0} \int_{-\infty}^{\infty} \delta_x(t-\tau)f(t)dt = f(\tau)$$

where $\delta_x(t)$ is the defining class of the regular sequence of well-behaved functions.

Using these relatively obscure concepts invented to permit mathematicans work with the generalized functions (the δ in particular) is useful in engineering, to enable easily computing the derivative of the δ function (the doublet, denoted $\delta'(t)$) and its integral (the unit step function $u(t)$). The approach enables determination that $\delta(t)$ is even ($\delta(t) = \delta(-t)$) and that

$$\delta(at) = \frac{1}{|a|}\delta(t).$$

Using the sifting property of the δ function is can be easily seen that $\delta(t) \longleftrightarrow 1$. Based on the symmetry property of the Fourier Transform, it follows that $1 \longleftrightarrow 2\pi\delta(\omega)$ (ω definition) or $1 \longleftrightarrow \delta(f)$ (f definition).

3.2 Fourier Transform of Periodic Signals as a Limit

The Fourier transform of a non-trival periodic function (as well as some others such as x^{-1}) does not, strictly speaking, exist because the function is not absolutely integrable. However, viewing the Fourier transform as the limit of a sequence for which the transform exists allows us to use the Fourier transform for many of these cases.

Consider the periodic function $f(t)$ multiplied by a function $g(t)$ which has $g_x(t) \longrightarrow 0$ as $t \longrightarrow \infty$, e.g., $g_x(t) = e^{-xt^2}$. The function $g_x(t)$ is choosen so that the Fourier transform $F_x(\omega)$ of $g_x(t)f(t)$ exists in the conventional sense. A sequence of transform pairs $g_x(t)f(t) \longleftrightarrow F_x(\omega)$ as $x \longrightarrow 0$ is then considered. If this limit exists,

$$\mathcal{F}\{f(t)\} = \lim_{x \longrightarrow 0} F_x(\omega).$$

This can be applied in the generalized function sense.

This limit method can be tricky to apply but is essential for computing the Fourier transform of many functions. For example, although sgn(t) is not periodic, the limit technique can be used to compute the Fourier transform of this function. First note that sgn(t) is not absolutely integrable. Let $g_x(t) = e^{-xt}$. Then,

$$g_x(t)f(t) = e^{-xt}u(t) - e^{xt}u(-t)$$

with corresponding $F_x(\omega)$ (using transform tables for $e^{xt}u(t)$)

$$F_x(\omega) = \frac{1}{x + j\omega} - \frac{1}{x - j\omega} \longleftrightarrow g_x(t)f(t) = e^{-xt}u(t) - e^{xt}u(-t)$$

In the limit as $x \longrightarrow 0$,

$$\mathcal{F}\{\text{sgn}(t)\} = \lim_{x \longrightarrow 0} F_x(\omega) = \frac{2}{j\omega}.$$

For the f definition form of the Fourier Transform,

$$\mathcal{F}\{\text{sgn}(t)\} = \lim_{x \longrightarrow 0} F_x(f) = \frac{1}{j\pi f}.$$

3.3 The Fourier Transform From the Fourier Series

The Fourier transform can be viewed as the limiting case of the Fourier Series as the period $T_0 \longrightarrow \infty$. Consider the case of a non-periodic function $f(t)$ which is assumed to be neglible for some $t > t_0$. Construct a periodic function $f_T(t)$ which consists of the sum of $f(t + nT)$. Ideally, $T > t_0$. $f_T(t)$ consists of periodic copies of $f(t)$ and has period T. Since $f_T(t)$ is periodic the Fourier series is computed as

$$f_T(t) = \sum_{n-\infty}^{\infty} F_n e^{jn\omega_0 t} \tag{3.1}$$

where $\omega_0 = 2\pi/T$ and

$$F_n = \frac{1}{T}\int_{-T/2}^{T/2} f_T(t)e^{-jn\omega_0 t}dt. \tag{3.2}$$

Note that write F_n can be written as $F_{n\omega_0}$.

As $T \longrightarrow \infty$, $\omega_0 \longrightarrow 0$. In the limit (assumed to exist), the spectrum is continuous and Eqs. (3.2) and (3.1) can be written as an integrals, i.e.,

$$F(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(t)e^{-j\omega t}dt$$

and

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)e^{-j\omega t}dt.$$

The 2π in this equation comes from the definition of ω_0.

3.4 The Fourier Series From the Fourier Transform

The Fourier Series is a special case of the Fourier transform. To show this note that (with the limits applied), recall

$$\begin{aligned} \delta(t-\tau) &\longleftrightarrow e^{-j\omega\tau} \\ e^{j\omega\tau} &\longleftrightarrow 2\pi\delta(\omega-\tau). \end{aligned}$$

From these relationships it follows that $\delta_N(t) \longleftrightarrow \Delta_N(\omega)$ where

$$\delta_N(t) = \sum_{n=-N}^{N} \delta(t-nT)$$

and

$$\Delta_N(\omega) = \sum_{n=-N}^{N} e^{jn\omega T} = \frac{\sin[(N+1/2)T\omega]}{\sin(T\omega/2)}.$$

In the limit at $N \longrightarrow \infty$, the transform pair,

$$S_\delta(t) \longleftrightarrow \omega_0 S_\delta(\omega)$$

is obtained where $\omega_0 = 2\pi/T$ and

$$\begin{aligned} S_\delta(t) &= \sum_{n=-\infty}^{\infty} \delta(t-nT) \\ S_\omega(t) &= \sum_{n=-\infty}^{\infty} \delta(\omega-n\omega_0). \end{aligned}$$

That $\delta_N(t) \longrightarrow S_\delta(t)$ is obvious. To see that $\Delta_N(\omega) \longrightarrow S_\omega(\omega)$ note that $\Delta_N(\omega)$ is a periodic function (in ω) with period ω_0. As $N \longrightarrow \infty$, in the interval $(-\omega/2, \omega/2)$, $\Delta_N(\omega)$ tends toward $\omega_0\delta(\omega)$.

Now consider a periodic function $f(t)$ with period T. Define $f_o(t)$ as,

$$f_o(t) = \begin{cases} f(t) & |t| \leq T/2 \\ 0 & \text{else.} \end{cases}$$

Then,

$$f(t) = \sum_{n=-\infty}^{\infty} f_o(t+nT) = f_o(t) * S_\delta(t).$$

Taking the transform both sides of this expression using the convolution theorem and the transform pair $S_\delta(t) \longleftrightarrow \omega_0 S_\delta(\omega)$,

$$F(\omega) = F_o(\omega) \cdot \omega_0 S_\omega(\omega) = \omega_0 F_o(\omega) \sum_{n=-\infty}^{\infty} \delta(\omega + n\omega_0).$$

Since $F_o(\omega)\delta(\omega - a) = F_o(a)\delta(\omega - a)$, it follows that

$$F(\omega) = \omega_0 \sum_{n=-\infty}^{\infty} F_o(n\omega)\delta(\omega + n\omega_0).$$

Thus, the Fourier transform consists of a weighted sum of δ functions at equally-spaced intervals. Taking the inverse transform of this expression (noting that $e^{j\omega\tau} \longleftrightarrow 2\pi\delta(\omega-\tau)$) the familiar Fourier series is obtained

$$f(t) = \sum_{n=-\infty}^{\infty} F_n e^{jn\omega_0 t}$$

where

$$F_n = \frac{1}{T} F_o(n\omega_0) = \frac{1}{T} \int_{-T/2}^{T/2} f(t) e^{-jn\omega_0 t} dt.$$

An interesting result of this derivation is that when computing F_n it may be simplier to compute $F_o(\omega)$ and sample it at $n\omega_0$ rather than directly computing F_n from it's definition.

3.5 Gibb's Phenomenon

The Fourier transform of signals with discontinuities can exhibit what is known as the *Gibb's Phenomenon*. To see this, consider the Fourier transform pair $f(t) \longleftrightarrow F(\omega)$ in which $f(t)$ has a finite discontinuity at the point $t = t_0$. Form the function $f_T(t)$ where

$$f_T(t) = \int_{-T}^{T} F(\omega) e^{j\omega t} d\omega.$$

Since $f(t) \longleftrightarrow F(\omega)$, with some manipulation it can be shown that

$$f_T(t) = \int_{-\infty}^{\infty} f(t) \frac{\sin T(t-\tau)}{\pi(t-\tau)} d\tau;$$

hence, $f_T(t)$ can be viewed as the convolution of $f(t)$ with the *Fourier-integral* kernal $\sin Tt/\pi t$. As $T \longrightarrow \infty$, $f_T(t) \longrightarrow f(t)$. However, at $t - t_0$ (the point of discontinuity), the limit of $f_T(t)$ is the *average* of the limit of $f(t)$ from the below and from above, i.e.,

$$\lim_{T\longrightarrow\infty} f_T(t) = \frac{1}{2} \left\{ \lim_{t\longrightarrow t_0^-} f(t) + \lim_{t_0^+ \longleftarrow t} f(t) \right\}.$$

Since, in this case, $f_T(t)$ is never close to $f(t)$ at $t = t_0$ for any T, $f_T(t)$ exhibits a high frequency oscillation as t approaches t_0. This oscillation is known as the *Gibb's phenomenon* as illustrated in Fig. 3.1.

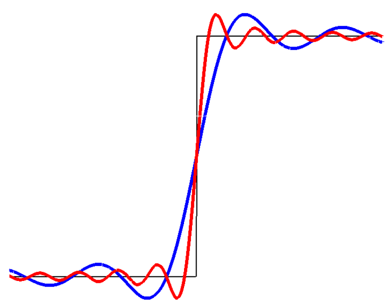


Figure 3.1: Illustration of Gibb's phenomenon. The black line shows the "true" function. The solid blue and dotted red lines show different levels of Fourier approximation for true curve based on the frequency cutoff (bandlimit) of the Fourier computation. Higher frequency cutoffs better approximate the function rise, but there is always overshoot and ringing.

Chapter 4

The Discrete Time Fourier Transform and The Discrete Fourier Transforms

While the Fourier transform can be directly applied to both continuous and discrete signals, special forms of the Fourier transform for discrete signals (sequences) are availble. These are the **Discrete-Time Fourier Transform** (DTFT), the **Discrete Fourier Series** (DFS), and the **Discrete Fourier Transform** (DFT). The DTFT can be applied to arbitrary discrete signals and is the discrete-time equivalent to the Fourier Transform (FT) while the DFS is the discrete-time equivalent of the Fourier Series (FS). The DFT is conceptually similar to the DFS but is often applied to aperiodic signals (even when it does not apply!). The relationship between the various transforms is summarized in Tab. 4.1.

Table 4.1: Relationship Between Signal Time and Frequency Domain Characteristics and Different Fourier Transforms

Time Domain | Frequency Domain

	Continuous	Discrete
Aperiodic	Fourier Transform Continuous, Aperiodic	DTFT Continuous, Periodic
Periodic	Fourier Series Discrete, Periodic	DFS / DFT Discrete, periodic

4.1 The Discrete Time Fourier Transform

The *Discrete Time Fourier Transform* (DTFT) is the general version of the Fourier transform designed for use with discrete signals (sequences). The DTFT of a discrete signal $x[n]$ is expressed as function of frequency ω as $X(e^{j\omega n})$ where

$$X(e^{j\omega n}) = \sum_{n=-\infty}^{\infty} x[n]e^{j\omega n}.$$

The DTFT is frequency written in this unusual form to emphasize the fact that $X(e^{j\omega n})$ is periodic (with period 2π) in ω. Some authors use Ω in place of ω to emphasize that the transform is applicable discrete signals. In this case Eq. (4.1) is written as

$$X(\Omega) = \sum_{n=-\infty}^{\infty} x[n]e^{j\Omega n}.$$

This latter notation is prefered in this text.

In effect, the DTFT maps a discrete signal (sequence) in the time domain, to a periodic signal in the frequency domain.

The inverse DTFT is,

$$x[n] = \int_{2\pi} X(\Omega)e^{j\Omega n} d\Omega.$$

Since $X(\Omega)$ is periodic in Ω with period 2π, the limits of the integration be chosen conveniently.

4.1.1 Derivation of the DTFT from the Fourier Transform

A discrete signal $x[n]$ can be modelled as a continuous signal $x_c(t)$ sampled at times $t = nT$ where T is the sample interval, i.e., $x[n] = x_c(t = nT)$. Note that $x_c(t = nT)$ can be written as

$$x_c(t = nT) = x(t)S_T(t)$$

where $S_T(t)$ is the sampling function defined in Chapter 1, i.e.,

$$S_T(t) = \sum_{k=-\infty}^{\infty} \delta(t - kT).$$

By decomposing the product $x(t)S_T(t)$ into the terms $x(t)$ and $S_T(t)$, noting that $S_T(T) \longleftrightarrow S_T(f)$ where

$$S_T(f) = \sum_{k=-\infty}^{\infty} \delta(f - 2\pi k/T),$$

and using the sifting property of the δ function, we can easily compute the Fourier transform $X_c(f)$ of $x(t)S_T(t)$ as

$$\begin{aligned}
\mathcal{F}\{x(t)S_T(t)\} &= \mathcal{F}\{x(t)\} * \mathcal{F}\{S_T(t)\} \\
&= X(f) * S_T(f) \\
&= \int_{-\infty}^{\infty} X(\alpha) S_T(f-\alpha) d\alpha \\
&= \int_{-\infty}^{\infty} X(\alpha) \sum_{k=-\infty}^{\infty} \delta(f-\alpha-2\pi k/T) d\alpha \\
&= \sum_{k=-\infty}^{\infty} \int_{-\infty}^{\infty} X(\alpha)\delta(f-\alpha-2\pi k/T) d\alpha \\
&= \sum_{k=-\infty}^{\infty} X(f-2\pi k/T)
\end{aligned}$$

4.1.2 The DTFT of Periodic Signals

Just as the Fourier transform can be defined in the limit so as to apply to periodic signals, the DTFT can be extended to apply to periodic signals. Given a discrete Fourier series (DFS) representation of a periodic signal (i.e., the DFS coefficients a_k), $X(\Omega)$ can be expressed as,

$$X(\Omega) = \sum_{k=-\infty}^{\infty} 2\pi a_k \delta\left(\Omega - \frac{2\pi k}{N}\right).$$

Given the DTFT of a periodic signal with period N, the DFS coefficients a_k are

$$a_k = \frac{1}{N} X(2\pi k/N).$$

4.2 The Discrete Fourier Series (DFS)

Coming...

4.3 The Discrete Fourier Transform (DFT)

The **Discrete Fourier Transform** (DFT) is a special version of the Fourier transform designed for use with *periodic* discrete signals (sequences). It is equivalent to the discrete Fourier seris. The DFT of a discrete signal $x[n]$ with a period is N, is expressed as $X[k]$ where

$$X[k] = \sum_{n=0}^{N-1} x[n] e^{-j2\pi nk/N}.$$

This is often written in terms of a *twiddle factor* W_N where

$$W_N = e^{-j2\pi/N}$$

as

$$X[k] = \sum_{n=0}^{N-1} x[n] W_N^{nk}.$$

The inverse DFT is[1]

$$x[n] = \frac{1}{N} \sum_{k=0}^{N-1} X[k] e^{j2\pi nk/N},$$

or, in terms of the twiddle factor,

$$x[n] = \frac{1}{N} \sum_{k=0}^{N-1} X[k] W_N^{-nk}.$$

In effect, the DFT maps a periodic, discrete signal (sequence) in the time domain, to a periodic, discrete signal (sequence) in the frequency domain. An efficient numerical implementations of the DFT is known as a **Fast Fourier Transform** (FFT). Such implementations employ the symmtry properties of the twiddle factor as it is raised to an integer power. For example, an N length DFT can be expressed in a radix 2 formulation as

$$X[k] = \sum_{n=0}^{N/2-1} x[2n] W_N^{2nk} + W_N^k \sum_{n=0}^{N/2-1} x[2n+1] W_N^{2nk}$$

as two smaller $(N/2)$ DFT formulations. The radix 3 formulation is

$$X[k] = \sum_{n=0}^{N/3-1} x[3n] W_N^{3nk} + W_N^k \sum_{n=0}^{N/3-1} x[3n+1] W_N^{3nk} + W_N^{2k} \sum_{n=0}^{N/3-1} x[3n+2] W_N^{3nk}.$$

The computational complexity of the DFT is $6N^2$ real operations, whereas an FFT approaches $5N \log_2 N - 3N$ real operations.

4.3.1 Derivation of the DFT from the DTFT

Coming...

4.3.2 Derivation of the DFT from the Fourier Series

Coming...

4.4 Transform Tables

This section includes tables of DTFT pairs.

[1]There is an alternate definition of the DFT and its inverse which involve leading scale factors of $1/\sqrt{N}$. However, these are not commonly used in Electrical Engineering.

4.4.1 Discrete Time Fourier Transform (DTFT) Tables

Definition and Key Properties

$$\begin{aligned} X(e^{j\omega n}) &= \sum_{n=-\infty}^{\infty} x[n]e^{j\omega n} \\ x[n] &= \int_{-\pi}^{\pi} X(e^{j\omega n})e^{j\omega n}d\omega \end{aligned}$$

Alternately, using Ω,

$$\begin{aligned} X(\Omega) &= \sum_{n=-\infty}^{\infty} x[n]e^{j\Omega n} \\ x[n] &= \int_{2\pi} X(e^{j\Omega n})e^{j\Omega n}d\Omega \end{aligned}$$

Key DTFT Properties

	$x[n] = \text{DTFT}^{-1}\{X(e^{j\omega})\}$	$\text{DTFT}\{x[n]\} = X(e^{j\omega})$
1	$x[n]$	$\int_{-\pi}^{\pi} X(\Omega)e^{j\Omega n}d\Omega$
2	$\sum_{n=-\infty}^{\infty} x[n]e^{j\Omega n}$	$X(\Omega)$
3	$ax[n] + by[n]$	$aX(\Omega) + bY(\Omega)$
4	$x[n + n_0]$	$e^{j\Omega N^0}X(\Omega)$
5	$x^*[n]$	$X^*(-\Omega)$
6	$x[-n]$	$X(-\Omega)$
7	$x[n] * y[n]$	$\frac{1}{2\pi}\int_{2\pi} X(\Omega')Y(\Omega - \Omega')d\Omega'$
8	$e^{j\Omega_0 n}x[n]$	$X(\Omega - \Omega_0)$
9	$x[n]y[n]$	$X(\Omega)Y(\Omega)$
10	$x[n] - x[n-1]$	$(1 - e^{-j\Omega})X(\Omega)$
11	$nx[n]$	$j\frac{d}{d\Omega}X(\Omega)$
12	$\sum_{k=-\infty}^{n} x[n]$	$\frac{1}{1 - e^{-j\Omega}}X(\Omega)\pi X(0)\sum_{k=-\infty}^{\infty} \delta(\Omega - 2\pi k)$
13	$x_{(k)}[n] = \begin{cases} x[n/k] & n \text{ multiple of } k \\ 0 & \text{else} \end{cases}$	$X(k\Omega)$
14	$\sum_{k=-\infty}^{\infty} \lvert x[n]\rvert^2$	$\frac{1}{2\pi}\int_{2\pi} X(\Omega')\lvert X(\Omega)\rvert^2 d\Omega$

Discrete Time Fourier Transform Table

Discrete Time Fourier Transform Table

	$x[n] = \text{DTFT}^{-1}\{X(\Omega)\}$	$X(\Omega) = \text{DTFT}\{x[n]\}$
1	$\delta[n]$	1
2	$\delta[n-n_0]$	$e^{-j\omega n_0}$
3	1	$2\pi \sum_{k=-\infty}^{\infty} \delta(\Omega - 2\pi k)$
4	$u[n]$	$\frac{1}{1-e^{-j\Omega}} + \pi \sum_{k=-\infty}^{\infty} \delta(\Omega - 2\pi k)$
5	$a^n u[n]$ $(\|a\| < 1)$	$\frac{1}{1-ae^{-j\Omega}}$
6	$(n+1)a^n u[n]$ $(\|a\| < 1)$	$\frac{1}{(1-ae^{-j\Omega})^2}$
7	$e^{j\Omega_0 n}$	$2\pi \sum_{k=-\infty}^{\infty} \delta(\Omega - \Omega_0 - 2\pi k)$
8	$\cos \Omega_0 n$	$\pi \sum_{k=-\infty}^{\infty} [\delta(\Omega - \Omega_0 - 2\pi k) + \delta(\Omega + \Omega_0 - 2\pi k)]$
9	$\sin \Omega_0 n$	$\frac{\pi}{j} \sum_{k=-\infty}^{\infty} [\delta(\Omega - \Omega_0 - 2\pi k) - \delta(\Omega + \Omega_0 - 2\pi k)]$
10	$\sum_{k=-\infty}^{\infty} \delta[n-kN]$	$\frac{2\pi}{N} \sum_{k=-\infty}^{\infty} \delta(\Omega - 2\pi k/N)$
11	$x[n] = \begin{cases} 1 & \|n\| < N \\ 0 & \text{else} \end{cases}$	$\frac{\sin[\Omega(2N+1)/2]}{\sin(\Omega/2)}$
12	$\frac{\sin \Omega_0 n}{\pi n}$	$\sum_{k=-\infty}^{\infty} \begin{cases} 1 & 0 \leq \|\Omega - 2\pi k\| \leq \Omega_0 \\ 0 & \text{else} \end{cases}$

4.4.2 Discrete Fourier Series (DFS) Tables

4.4.3 Definition and Key Properties

$$
\begin{aligned}
a_k &= \sum_{n\in[N]}^{\text{Period}=N} x[n]e^{-j2\pi nk/N} \\
x[n] &= \frac{1}{N} \sum_{k\in[N]} a_k e^{j2\pi nk/N}
\end{aligned}
$$

Key Discrete Fourier Series Properties

	$x[n] = \text{DFT}^{-1}\{X[k]\}$	$\text{DFT}\{x[n]\} = X[k]$
1	$x[n]$	$\sum_{n=0}^{N-1} x[n]e^{-j2\pi nk/N}$
2	$\frac{1}{N}\sum_{k=0}^{N-1} X[k]e^{j2\pi nk/N}$	$X[k]$
3	$ax[n] + by[n]$	$aX[k] + bY[k]$
4	$x[n + n_0]$	$e^{-j2\pi kn_0/N}X[k] = W_N^{-kn_0}X[k]$
5	$x^*[n]$	$X^*[[-k]]_N$
6	$Nx[n]y[n]$	$\sum_{p=0}^{N-1} X[p]Y[k-p]$
7	$\sum_{p=0}^{N-1} x[p]y[n-p]$	$X[k]y[k]$
8	$e^{-j2\pi nk_0/N}x[n]$	$X[[k-k_0]]_N$

where $X[[k]]_N$ is the N-point periodic extension of $X[k]$, i.e., $X[p]$ where $p = k \bmod N$.

Discrete Fourier Series Table

Discrete Fourier Series Table

	$x[n] = \text{DFS}^{-1}\{a_k\}$	$\text{DFS}\{x[n]\} = a_k$
1	$\delta[n]$	1
2	N	$\delta[k]$
3	$\delta[n - n_0]$	$e^{-j2\pi kn_0/N}$
4	$\cos k\pi n/N$	$a_k = a_{-k} = 1/2$
6	$\sin k\pi n/N$	$a_k = -j/2$, $a_{-k} = j/2$

4.4.4 Discrete Fourier Transform (DFT) Tables

4.4.5 Definition and Key Properties

$$\begin{aligned} X[k] &= \sum_{n=0}^{N-1} x[n]e^{-j2\pi nk/N} = \sum_{n=0}^{N-1} x(n)W_N^{nk} \\ x[n] &= \frac{1}{N}\sum_{k=0}^{N-1} X[k]e^{j2\pi nk/N} = \frac{1}{N}\sum_{k=0}^{N-1} X[k]W_N^{-nk} \\ & W_N = e^{-j2\pi/N} \end{aligned}$$

Key DFT Properties

	$x[n] = \text{DFT}^{-1}\{X[k]\}$	$\text{DFT}\{x[n]\} = X[k]$
1	$x[n]$	$\sum_{n=0}^{N-1} x[n]e^{-j2\pi nk/N} = \sum_{n=0}^{N-1} x[n]W_N^{-nk}$
2	$\frac{1}{N}\sum_{k=0}^{N-1} X[k]e^{j2\pi nk/N} = \frac{1}{N}\sum_{k=0}^{N-1} X[k]W_N^{-nk}$	$X[k]$
3	$ax[n] + by[n]$	$aX[k] + bY[k]$
4	$x[n+n_0]$	$e^{-j2\pi kn_0/N}X[k] = W_N^{-kn_0}X[k]$
5	$x^*[n]$	$X^*[[-k]]_N$
6	$Nx[n]y[n]$	$\sum_{p=0}^{N-1} X[p]Y[k-p]$
7	$\sum_{p=0}^{N-1} x[p]y[n-p]$	$X[k]y[k]$
8	$e^{-j2\pi nk_0/N}x[n] = W_N^{nk_0}x[n]$	$X[[k-k_0]]_N$

where $X[[k]]_N$ is the N-point periodic extension of $X[k]$, i.e., $X[p]$ where $p = k \bmod N$.

Discrete Fourier Transform Table

See also the Discrete Fourier Series Table.

Discrete Fourier Transform Table

	$x[n] = \text{DFT}^{-1}\{X[k]\}$	$\text{DFT}\{x[n]\} = X[k]$
1	$\delta[n]$	1
2	N	$\delta[k]$
3	$\delta[n-n_0]$	$e^{-j2\pi kn_0/N} = W_N^{-kn_0}$

Chapter 5

The Fourier Series and Orthogonal Transforms

5.1 Background

The Fourier transform can be applied to virtually any signal. The **Fourier Series** is a special case of the Fourier transform which can be used when the signal is *periodic*. The basic idea of the Fourier series, first put forth by Joseph Fourier, is that a periodic function with period T_0 could be described by a weighted sum of cosine and sine functions, i.e.,

$$f(t) = \sum_{n=0}^{\infty} (a_n \cos n\omega_0 t + b_n \sin n\omega_0 t)$$

where $\omega_0 = 2\pi/T_0$. This idea is based on the more general concept on an *orthogonal transform.*

5.2 Fourier Series

The **Fourier Series** (FS) is an orthogonal transform using Euler functions. The Fourier series can be derived from the Fourier Transform or visa versa (see the next Chapter). The orthogonal functions used in the Fourier Series are $\cos n\omega_0 t$ and $\sin n\omega_0 t$ or $e^{-jn\omega_0 t}$ where $\omega_0 = 2\pi T_0$.

With only a little effort it can shown that

$$\begin{aligned}
\langle \cos m\omega_0 t, \cos n\omega_0 t \rangle &= \int_{t_0}^{t_0+T_0} \cos m\omega_0 t \cos n\omega_0 t dt &= \begin{cases} 0 & \text{if } m \neq n, \\ 1 & \text{if } m = n \end{cases} \quad \forall m, n \geq 0 \\
\langle \cos m\omega_0 t, \sin n\omega_0 t \rangle &= \int_{t_0}^{t_0+T_0} \cos m\omega_0 t \sin n\omega_0 t dt &= 0 \quad \forall m, n \geq 0 \\
\langle \sin m\omega_0 t, \sin n\omega_0 t \rangle &= \int_{t_0}^{t_0+T_0} \sin m\omega_0 t \sin n\omega_0 t dt &= \begin{cases} 0 & \text{if } m \neq n, \\ 1 & \text{if } m = n \end{cases} \quad \forall m, n \geq 0
\end{aligned}$$

and that

$$\langle e^{-jm\omega_0 t}, e^{-jn\omega_0 t}\rangle = \int_{t_0}^{t_0+T_0} e^{-jm\omega_0 t} e^{-jn\omega_0 t} dt = \begin{cases} 0 & \text{if } m \neq n, \\ 1 & \text{if } m = n \end{cases} \quad \forall m, n.$$

Hence, the Euler functions form an orthonormal set.

Probably the most common form of the Fourier series is,

$$f(t) = a_0 + \sum_{n=1}^{\infty} \{a_n \cos n\omega_o t + b_n \sin n\omega_0 t\}$$

where a_n and b_n are,

$$\begin{aligned} a_0 &= \frac{1}{T_0} \int_{t_0}^{t_0+T_0} f(t) dt \\ a_n &= \frac{2}{T_0} \int_{t_0}^{t_0+T_0} f(t) \cos n\omega_0 t dt \quad \forall n > 1 \\ b_n &= \frac{2}{T_0} \int_{t_0}^{t_0+T_0} f(t) \sin n\omega_0 t dt \quad \forall n > 1 \end{aligned}$$

and $\omega_0 = 2\pi/T_0$.

The Fourier Series can also be written as,

$$f(t) = a_0 + \sum_{n=1}^{\infty} C_n \cos(n\omega_o t + \theta_n)$$

where C_n and θ_n are,

$$\begin{aligned} C_n &= \sqrt{a_n^2 + b_n^2} \\ \theta_n &= \arctan b_n/a_n \end{aligned}$$

where a_n and b_n are defined above.

An alternate form (known as the *exponential Fourier Series*) is defined as

$$f(t) = \sum_{n=-\infty}^{\infty} F_n e^{jn\omega_0 t}$$

where $\omega_0 = 2\pi f_0$ and $T_0 = 1/f_0$ is the fundamental period. The fundamental period is the minimum value of T_0 for which $g(t) = g(t + T_0)$ for all t. F_n (which we could write as $F(n\omega_0)$) is

$$F_n = \frac{1}{T_0} \int_{t_0}^{t_0+T_0} f(t) e^{-jn\omega_0 t} dt$$

where t_0 is arbitrary.

F_n and a_n and b_n are related by the following:

$$\begin{aligned} a_0 &= F_0 \\ a_n &= F_n + F_{-n} \quad n \neq 0 \\ b_n &= j(F_n - F_{-n}) \quad n \neq 0 \\ F_n &= \frac{1}{2}(a_n - jb_n) \quad n \geq 1 \\ F_{-n} &= \frac{1}{2}(a_n + jb_n) \quad n \geq 1 \end{aligned}$$

Note that $F_n = F^*_{-n}$.

As shown in the next Chapter, it follows from the definition of the Fourier series that the Fourier transform of a periodic signal $f(t)$ with Fourier series F_n, is

$$\mathcal{F}\{f(t)\} = 2\pi \sum_{n=-\infty}^{\infty} F_n \delta(\omega - n\omega_0).$$

5.2.1 Properties of the Fourier Series

The properties derived for the Fourier transform apply to the Fourier series.

Note that when the signal is even, the sine terms of the Fourier series are zero while if the signal is odd, the cosine terms are zero.

For two signals $f(t)$ and $g(t)$ with Fourier series F_n and G_n with the same fundamental period, the *convolution* of $f(t)$ and $g(t)$ has a Fourier series $F_n G_n$.

Parseval's formula becomes

$$\frac{1}{T}\int_t^{t+T_0} f(t)g^*(t)dt = \sum_{n=-\infty}^{\infty} F_n G_n^*.$$

5.2.2 Gibbs Phenomenon

The Fourier series exhibits the Gibb's phenomenon (see Chapter 3). If there is a discontinuity in the signal at a point $t = a$, the Fourier series will attempt converge to a point midway between the left and right limits at $t \longrightarrow a$.

5.3 Fourier Series Transform Tables

5.3.1 Definition and Key Properties

$$f(t) = \sum_{n=1}^{\infty}(a_n \cos n\omega_0 t + b_n \sin n\omega_0 t) \quad \text{(Trig. form)}$$

$$a_0 = \frac{1}{T_0}\int_{t_0}^{t_0+T_0} f(t)dt$$

$$a_n = \frac{2}{T_0}\int_{t_0}^{t_0+T_0} f(t)\cos n\omega_0 t dt \quad \forall n > 1$$

$$b_n = \frac{2}{T_0}\int_{t_0}^{t_0+T_0} f(t)\sin n\omega_0 t dt \quad \forall n > 1$$

$$f(t) = \sum_{n=-\infty}^{\infty} F_n e^{jn\omega_0 t} \quad \text{(Exponential form)}$$

$$F_n = \frac{1}{T_0}\int_{t_0}^{t_0+T_0} f(t)e^{-jn\omega_0 t}dt$$

Coefficient relationships

$$a_0 = F_0$$

$$a_n = F_n + F_{-n} \quad n \neq 0$$

$$b_n = j(F_n - F_{-n}) \quad n \neq 0$$

$$F_n = \frac{1}{2}(a_n - jb_n) \quad n \geq 1$$

$$F_{-n} = \frac{1}{2}(a_n + jb_n) \quad n \geq 1$$

Key Properties of the Fourier Series

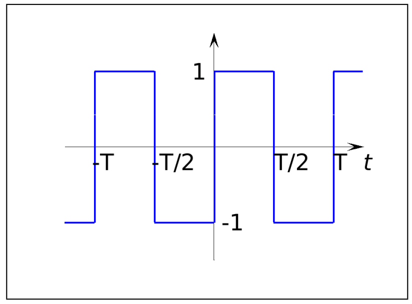
Period$= T$, $\omega_0 = 2\pi/T$

	$x(t)$	Fourier Series Coefficients
	$x(t)$	$\sum_{n=\infty}^{\infty} F_n e^{jn\omega_0}$
	$y(t)$	$\sum_{n=\infty}^{\infty} G_n e^{jn\omega_0}$
1	$x(t-\tau)$	$F_n e^{-jn\omega_0\tau}$
2	$e^{jM\omega_0 t}x(t)$ (M integer)	F_{n-M}
3	$x^*(t)$	F^*_{-n}
4	$x(-t)$	F_{-n}
5	$x(\alpha t)$ (period=T/α)	F_n
6	$x(t)y(t)$	$\sum_{k=-\infty}^{\infty} F_n G_n$
7	$x(t) * y(t) = \int_T x(\tau)y(t-tau)d\tau$	TF_nG_n
7	$Even\{x(t)\}$	$Re\{F_n\}$
8	$Odd\{x(t)\}$	$jIm\{F_n\}$
10	$\frac{d}{dt}x(t)$	$jn\omega_0 F_n$
11	$\int_{-\infty}^{t} x(t)dt$ ($F_0 = 0$)	$F_n/(jn\omega_0)$
12	$\frac{1}{T}\int_T \lvert x(t)\rvert^2 dt$	$\sum_{n=-\infty}^{\infty} \lvert F_n\rvert^2$

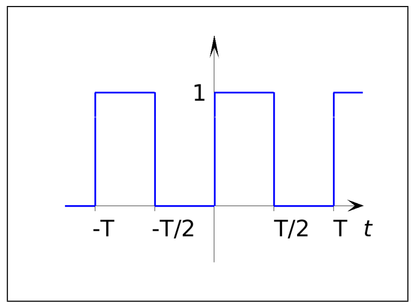
5.3.2 Pictitorial Fourier Series Transforms

In the following T is the period and $\omega_0 = 2\pi/T$. All sums are over positive n.

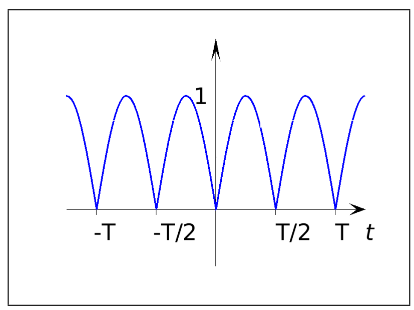
$$f(t) = \frac{4}{\pi} \sum_{n\ odd} \frac{1}{n} \sin n\omega_0 t$$



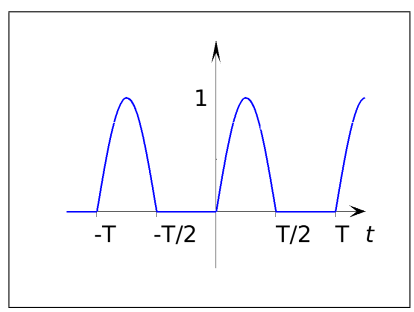
$$f(t) = \frac{1}{2} + \frac{2}{\pi} \sum_{n\ odd} \frac{1}{n} \sin n\omega_0 t$$



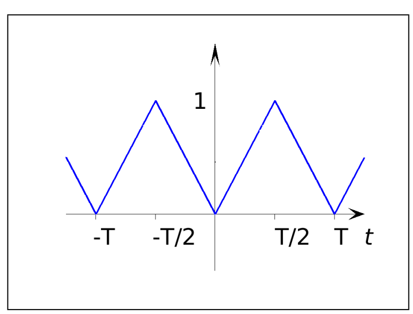
$$f(t) = \frac{2}{\pi} - \frac{4}{\pi} \sum_{n\ even} \frac{1}{n^2 - 1} \cos n\omega_0 t$$



$$f(t) = \frac{1}{\pi} + \sin \omega_0 t - \frac{2}{\pi} \sum_{n\ even} \frac{1}{n^2 - 1} \cos n\omega_0 t$$



$$f(t) = \frac{1}{2} - \frac{4}{\pi^2} \sum_{n\ odd} \frac{1}{n^2} \cos n\omega t$$



5.3.3 Fourier Series Transform Table

Period$= T$, $\omega_0 = 2\pi/T$

	Signal over 1 period	Fourier Series Coefficients
	$g(t)$	$\sum_{k=0}^{\infty}(a_k \cos k\omega_0 t + b_k \sin k\omega_0 t)$ $a_k = \int_{\text{one period}} f(t)\cos k\omega_0 t dt$ $b_k = \int_{\text{one period}} f(t)\sin k\omega_0 t dt$ $\sum_{k=\infty}^{\infty} c_k e^{jk\omega_0}$ $c_k = \int_{\text{one period}} f(t)e^{-jk\omega_0 t}dt$
1	$f(t) = \begin{cases} t & \lvert t\rvert \leq T/2 \\ 0 & \text{else} \end{cases}$	$a_k = 0 \quad b_k = \frac{T}{2\pi n}(-1)^{n+1}$
2	$f(t) = \begin{cases} t^2 & \lvert t\rvert \leq T/2 \\ 0 & \text{else} \end{cases}$	$a_0 = \frac{T^2}{6} \quad a_k = \frac{T^2}{n^2\pi^2}(-1)^n \quad b_k = 0$
3	$f(t) = \begin{cases} t^3 & \lvert t\rvert \leq T/2 \\ 0 & \text{else} \end{cases}$	$a_k = 0 \quad b_k = \frac{T^3}{4\pi n}(-1)^{n+1}\left(1 - \frac{6}{n^2\pi^2}\right)$
4	$f(t) = \begin{cases} e^{a\lvert t\rvert} & \lvert t\rvert \leq \pi \\ 0 & \text{else} \end{cases}$	$a_k = \frac{2a[1+(-1)^{n+1}e^{-a\pi}]}{\pi(n^2 - a^2}$ $\quad b_k = 0 \quad T = \pi$
5	$f(t) = \begin{cases} t & \lvert t\rvert \leq T_1 \\ 0 & \text{else} \end{cases}$	$a_0 = \frac{2T_1}{T} \quad a_k = \frac{\sin k\omega_0 T_1}{\pi k} \quad b_k = 0$

5.4 Orthogonal Transforms

The Fourier series is a special case of the more general class of orthogonal transforms. In this section a general othogonal transform is defined and two additional orthogonal transforms are considered.

Define $\langle f, g\rangle$ in the continuous case as,

$$\langle f, g\rangle = \int_L^H f(x)g(x)w(x)dx$$

where $w(x)$ is a weighting function defined over the interval $\mathcal{I} = [L, H]$ (L and H may be infinite). A family of functions $\{\phi_n,\ n \in \mathcal{I}_d\}$ will be *orthogonal* with respect to the weighting function $w(x)$ over $[L, H]$ if and only if,

$$\langle \phi_m \phi_n\rangle = \begin{cases} 0 & if m \neq n, \\ \lambda_m \quad (\lambda_m \neq 0) & if m = n. \end{cases} \quad \forall m, n \in \mathcal{I}_d$$

The set $\mathcal{I}_d$ may be $[-\infty, \infty]$ or $[0, \infty]$. $\{\phi_n\}$ will be *orthonormal* if $\lambda_m = 1$ for all m.

In the discrete case, $\langle f, g\rangle$ is defined as,

$$\langle f, g\rangle = \sum_{i=0}^{N} f(x_i)g(x_i)w(x_i) \quad x_i \neq x_j \forall i, j.$$

Given a spanning set of orthogonal functions $\{\phi_n\}$ defined over an interval $\mathcal{I}$, an orthogonal transform allows us to write a function $f(t)$, $t \in \mathcal{I}$ as,

$$f(t) = \sum_{n \in \mathcal{I}_d} a_n \phi_n(t)$$

where

$$a_n = \int_{\tau \in \mathcal{I}} f(t)\phi_n(t)w(t)dt.$$

Examples of continuous orthogonal functions commonly used in orthogonal transforms include the Euler functions (sine, cosine, and complex exponentials) and polynomials such as Legendre, Chebychev, Laguere, and Hermit. For the discrete case, examples include Walsh and Rademaker functions and Chebychev polynomials.

5.5 Orthogonal Polynomials

The Fourier series is a particular example of an orthogonal transform. For reference, the following sections give the properties of two important families of orthogonal polynomials.

5.5.1 Chebychev Polynomials

Chebychev[1] polynomials exhibit both continuous and discrete orthogonality. Chebychev polynomials are defined over the interval $I = [-1, 1]$. To apply Chebychev polynomials in terms of the parameter t over the interval $[a, b]$, make the substitution,

$$t = \frac{1}{2}(a+b) + \frac{1}{2}(b-a)x, \quad t \in [a,b] \Longleftrightarrow x \in [-1,1]$$

An n^{th} order Chebychev polynomial of the first kind is defined as

$$T_n(x) = \cos n\theta \quad \theta = \arccos x$$

where $x \in I = [-1, 1]$. (Note: for $x \geq 1$, the definition $T_n(x) = \cosh nt$ where $x = \cosh t$, $t \geq 0$, may be used). Chebychev polynomials of the first kind satisfy the recussion relation,

$$\begin{aligned} T_0(x) &= 1 \\ T_1 &= x \\ T_{n+1}(x) &= 2xT_n(x) - T_{n-1}(x) \quad n \geq 1 \end{aligned}$$

with the general formula,

$$T_n(x) = \frac{n}{2} \sum_{i=0}^{[[n/2]]} (-1)^i \frac{(n-i-1)!}{i!(n-2i)!} (2x)^{n-2i}$$

[1] Also spelled as Tchebycheff

Table 5.1: The Coefficients of Cheychev polynomials of the first kind

	1	x	x^2	x^3	x^4	x^5	x^6	x^7	x^8
$T_0(x)$	1								
$T_1(x)$		1							
$T_2(x)$	-1		2						
$T_3(x)$		-3		4					
$T_4(x)$	1		-8		8				
$T_5(x)$		5		-20		16			
$T_6(x)$	-1		18		-48		32		
$T_7(x)$		-7		56		-112		64	
$T_8(x)$	1		-32		160		-256		128

for $n = 1, 2, \ldots$ with $T_0(x) = 1$ and with

$$[[n/2]] = \begin{cases} n/2 & n\text{even} \\ (n-1)/2 & n\text{odd.} \end{cases}$$

An n^{th} order Chebychev polynomial of the second kind is defined as

$$U_n(x) = \frac{\sin(n+1)\theta}{\sin\theta} \qquad \theta = \arccos x.$$

Note that $\sin(\arccos x) = \sqrt{1-x^2}$. Chebychev polynomials of the second kind satisfy the recursion relation,

$$\begin{aligned} U_0(x) &= 1 \\ U_1(x) &= 2x \\ U_n(x) &= 2T_n(x) + U_{n-2}(x) \quad n \geq 2 \end{aligned}$$

The first few Chebychev polynomials are given in Tables 5.5.1 and 5.5.1. Chebychev polynomials are special cases of the *hypergeometric function*

$$T_n(x) = F(-n, n, \frac{1}{2}; \frac{1}{2} - \frac{x}{2}).$$

Key properties of Chebychev polynomials include:

1. The Chebychev polynomials are continuous and finite on $I = [-1, 1]$.

2. On the interval $x \in I = [-1, 1]$, $|T_n(x)| \leq 1$ for all n. The points $\eta_i^{(n)} = \cos\frac{i\pi}{n}$, $i = 0, 1, \ldots, n$, in $I = [-1, 1]$ at which $|T_n(x)| = 1$ are known as the *extrema* of $T_n(x)$.

3. The n roots $\zeta_i^{(n)} = \cos\frac{(2i-1)}{n}\frac{\pi}{2}$, $i = 1, \ldots, n$, of the Chebychev polynomial $T_n(x)$, $n > 1$, are all (a) simple, (b) real, and (b) lie in the interval $I = [-1, 1]$.

Table 5.2: The coefficients of Cheychev polynomials of the second kind

	1	x	x^2	x^3	x^4	x^5	x^6	x^7	x^8
$U_0(x)$	1								
$U_1(x)$		2							
$U_2(x)$	-1		4						
$U_3(x)$		-4		8					
$U_4(x)$	1		-12		16				
$U_5(x)$		6		-32		32			
$U_6(x)$	-1		24		-80		64		
$U_7(x)$		-8		80		-192		128	
$U_8(x)$	1		-40		240		-448		256

4. The Chebychev polynomials $\{T_n(x)\}_{n=0}^{\infty}$ form a sequence of orthogonal polynomials on $I = [-1, 1]$ with respect to the weight function $w(x) = \sqrt{1 - x^2}$,

$$\int_{-1}^{1} T_k(x)T_m(x)w(x)dx = \begin{cases} 0, & m \neq k, \\ \frac{\pi}{2}, & k \neq 0, \\ \pi, & k = 0 \end{cases}$$

5. The Chebychev polynomials are orthogonal on a finite point sets in I where the weight function is 1: if $\zeta_1, \ldots, \zeta_n$ are the zeros of $T_n(x)$ then,

$$\sum_{i=1}^{n} T_k(\zeta_i)T_m(\zeta_i) = \begin{cases} \frac{(-1)^p+(-1)^q}{2}n, & \text{if } k+m = 2pn \text{and} |k-m| = 2qn \\ (-1)^2\frac{n}{2}, & \text{if } k+m = 2sn \text{and} |k-m| \neq 2rn \\ & \text{or} |k-m| = 2sn \text{and} k+m \neq 2rn, \\ 0, & \text{otherwise.} \end{cases}$$

6. Symmetry property: $T_n(-x) = (-1)^n T_n(x)$.

7. The leading coefficient of $T_n(x)$ is 2^{n-1} for $n \geq 1$ and 1 for $n = 0$.

8. Chebychev polynomials satisfy the *minimax property*. Of all nth-degree polynomials with leading coefficient 1, the polynomial $2^{1-n}T_n(x)$ has the smallest maximum norm (of 2^{1-n}) in $[-1, 1]$.

9. The leading coefficient of $U_n(x) = u_n x^n + \cdots + u_1 x + u_0$ is 2^n and

$$U_n(x) = \sum_{i=0}^{n} x^i T_{n-i}(x).$$

Some useful Chebychev identities

1. $T_n'(x) = nU_{n-1}(x)$
2. $U_n(x) = U_{n-2}(x) = 2T_n(x)$
3. $T_n(x) = U_n(x) - xU_{n-1}(x)$
4. $\frac{1}{2}U_{2k}(x) = \frac{1}{2} + T_2(x) + T_4(x) + \cdots + T_{2k}(x) \quad k \geq 0$
5. $\frac{1}{2}U_{2k+1}(x) = T_1(x) + T_3(x) + \cdots + T_{2k+1}(x) \quad k \geq 0$
6. $x\frac{1}{2}U_{2k-1}(x) = 1 + 2T_2(x) + 2T_4(x) + \cdots + T_{2k-2}(x) \quad k \geq 0$
7. $U_{nm-1}(x) = U_{m-1}(T_n(x))U_{n-1}(x)$
8. $T_m(x)T_n(x) = \frac{1}{2}(T_{m+n}(x) + T_{|m-n|}(x)) \quad m \geq 0,\ n \geq 0$
9. $T_m(T_n(x)) = T_{mn}(x) \quad m \geq 0,\ n \geq 0$
10. $(T_{m+n}(x) - 1)(T_{|m-n|}(x) - 1) = (T_m(x) - T_n(x))^2$
11. $\displaystyle\int T_n(x)dx = \frac{1}{2}[T_{n+1}(x)/(n+1) - T_{n-1}(x)/(n-1)] + Const, \quad n \geq 2$

5.5.2 Legendre Polynomials

Legendre polynomials are defined over the interval $I = [-1, 1]$. To apply Legendre polynomials in terms of the parameter θ over the interval $[a, b]$, make the substitution,

$$\theta = \frac{1}{2}(a+b) + \frac{1}{2}(b-a)x, \quad t \in [a, b] \Longleftrightarrow x \in [-1, +1]$$

Legendre polynomials are defined as

$$P_0(x) = 1, \qquad P_n(x) = \frac{1}{2^n n!}\frac{d^n}{dx^n}[(x^2-1)^n], \ \ n > 0$$

where $x \in I = [-1, 1]$. The weighting function is $w(x) = 1$. Legendre polynomials satisfy the recursion relation,

$$\begin{aligned} P_0(x) &= 1 \\ P_1 &= x \\ P_2 &= \frac{1}{2}(3x^2 - 1) \\ P_{n+1}(x) &= \frac{2n+1}{n+1}xP_n(x) - \frac{n}{n+1}P_{n-1}(x) \quad n \geq 1. \end{aligned}$$

The first few Legendre polynomials are given in Table 5.5.2.

Key properties of Legendre polynomials include:

1. The Legendre polynomials are continuous and finite on $I = [-1, 1]$ with $|P_n(x)| \leq 1$ for $x \in [-1, 1]$.

2. The Legendre polynomials $\{P_n(x)\}_{n=0}^{\infty}$ form a sequence of orthogonal polynomials on $I = [-1, 1]$ with respect to the weight function $w(x) = 1$,

$$\int_{-1}^{1} P_k(x)P_m(x) = \begin{cases} 0, & m \neq k, \\ \frac{2}{2k+1}, & \text{if } m = k \end{cases}$$

Table 5.3: Scaled Legendre polynomial coefficients

	1	x	x^2	x^3	x^4	x^5	x^6	$2^n n!$
$2^n n! P_0(x)$	1							1
$2^n n! P_1(x)$		2						2
$2^n n! P_2(x)$	-4		12					8
$2^n n! P_3(x)$		-72		120				48
$2^n n! P_4(x)$	144		-1440		1680			384
$2^n n! P_5(x)$		7200		-33600		30240		3840
$2^n n! P_6(x)$	-14400		302400		-907200		665280	46080

3. Symmetry property: $P_n(-x) = (-1)^n P_n(x)$.

4. The leading coefficient p_n of $P_n(x)$ is,

$$\frac{1}{2^n n!} 2n(2n-1)(2n-2)\ldots(n+1).$$

Chapter 6

The Laplace and $\mathcal{Z}$ Transforms

The Laplace and $\mathcal{Z}$ transforms represent important generalizations of the Fourier transform. Unlike the Fourier transform, the Laplace transform can be applied to continuous unstable signals and systems. Similarly, the $\mathcal{Z}$ transform is designed for application with discrete signals and systems. These transforms, along with the discrete-time Laplace transform, are considered in this Chapter. Other transforms related to the Fourier transform are considered in the next Chapter. This Chapter also contains a brief overview of partial fraction expansion which is widely used in computing the inverse Laplace and $\mathcal{Z}$ transforms.

6.1 The Laplace Transform

The Laplace transform finds most frequent application in continuous linear differential equation systems. The Laplace transform takes three general forms. The most general form is the **bilateral** or **two-sided Laplace transform**. The **unilateral** or **one-sided Laplace transform** may take two forms: the unilateral (one-sided) **zero-postive Laplace transform** which is the most common, and the unilateral (one-sided) **zero-negative Laplace transform**. The unilateral transform can be defined in terms of the bilateral Laplace transform of a signal multiplied by $u(t)$.

While these versions of the Laplace transform are for continuous signals and systems, the **discrete-time Laplace transform**, sometimes referred to as the **starred Laplace transform**, is also defined for discrete signals and systems. It is occasionally used in control systems though the $\mathcal{Z}$ transform is more widely used because of its compactness. The discrete-time Laplace transform is described in a separate subsection.

The Laplace transform variable is $s = \sigma + j\omega$ where $\sigma \geq 0$ and ω are generally real. The unilateral (one-sided) zero-positive Laplace transform $F_+(s) = \mathcal{L}_+\{f(t)\}$ of the signal $f(t)$ is defined as,

$$F_+(s) = \mathcal{L}_+\{f(t)\} = \int_{0^+}^{\infty} f(t)e^{-st}dt$$

while the unilateral (one-sided) zero-negative Laplace transform $F_-(s) = \mathcal{L}_-\{f(t)\}$ is defined as,

$$F_-(s) = \mathcal{L}_-\{f(t)\} = \int_{0^-}^{\infty} f(t)e^{-st}dt$$

(if the integrals exist). The bilateral (two-sided) Laplace transform $F(s) = \mathcal{L}\{f(t)\}$ is defined as,

$$F(s) = \mathcal{L}\{f(t)\} = \int_{0^+}^{\infty} f(t)e^{-st}dt.$$

Note that if $f(t) = 0$ for all $t \leq 0$ with no singularities (δ functions or higher-order δ functions) at the origin, the three versions of the Laplace transform are equivalent.

In the definitions of the unilateral Laplace transforms the lower limit (0^+) is used to denote that the limit is from above, i.e.,

$$F_+(s) = \int_{0^+}^{\infty} f(t)e^{-st}dt = \lim_{h \longrightarrow 0^+,\ h>0} \int_{h}^{\infty} f(t)e^{-st}dt$$

while the lower limit (0^-) is used to denote that the limit is from below, i.e.,

$$F_-(s) = \int_{0^-}^{\infty} f(t)e^{-st}dt = \lim_{h \longrightarrow 0^-,\ h<0} \int_{h}^{\infty} f(t)e^{-st}dt.$$

These distinctions may be critical when there are discontinuities in $f(t)$ at $t = 0$. The unilateral zero-positive Laplace transform is primarily used for initial condition problems with linear differential equations.

The Laplace transform only exists if the defining integrals are less than infinity, i.e., the one-sided zero-positive Laplace transform exists if and only if,

$$\int_{0^+}^{\infty} f(t)e^{-st}dt < \infty$$

while the one-sided zero-negative Laplace transform exists if and only if,

$$\int_{0^-}^{\infty} f(t)e^{-st}dt < \infty$$

and the two-sided Laplace transform exists if and only if,

$$\int_{-\infty}^{\infty} f(t)e^{-st}dt < \infty.$$

In general, these conditions apply only over a **Region of Convergence** consisting of a band of values in the s plane, e.g., the region of convergence $R_- \leq Re[s] \leq R_+$ where R_- and $< R_+$ are real constants which may be infinite.

Frequently, the Laplace transform $F(s)$ of a signal or system can be expressed as a *rational polynomials*, e.g.,

$$F(s) = \frac{\displaystyle\sum_{n=0}^{N} b_n s^n}{\displaystyle\sum_{k=0}^{D} a_k s^k} = \frac{N(s)}{D(s)}.$$

The values of s corresponging to the roots of the denominator polynomial $D(s)$ are known as the *poles* of the transform while the roots of the numerator polynomial $N(s)$ are the *zeros* of the transform. These are frequently mapped on a s-plane *pole-zero map*. On the pole-zero map the region of convergence is a band delimited by poles. The region of convergence does NOT contain any poles. For a given pole-zero map it may be possible to define several possible regions of convergence. Each distinct region of convergence corresponds to a separate time-domain signal. Note that a *causal* signal or system has a region of convergence to the right of the right-most pole. However, just because the region of convergence is to the right of the right-most pole does not guarantee that the signal is causal.

The **inverse Laplace transform** $f(t) = \mathcal{L}^{-1}\{F(s)\}$ (if it exists) is given by

$$f(t) = \frac{1}{2\pi j} \int_{\sigma - j\infty}^{\sigma + j\infty} F(s) e^{st} ds$$

where $\sigma > 0$ is a real constant within the region of convergence. The inverse Laplace transform can be computed using tables, directly evaluating this integral, or, when the Laplace transform is a rational polynomial, using partial fraction expansion and inverse transforms from the table.

6.1.1 Relation of the Laplace and Fourier Transforms

The Laplace transform represents a generalization of the Fourier transform where the real-valued ω in the Fourier transform is replaced with complex-valued s. If the $j\omega$ axis is contained within the region of convergence of the transform, the Fourier transform may be computed from the Laplace transform by replacing s with $j\omega$, i.e.,

$$\mathcal{F}\{f(t)\} = F(s)\Big|_{s=j\omega}.$$

If the $j\omega$ axis is contained within the region of convergence, the existence of the Fourier transform is assured and the corresponding time-domain system is *stable*.

The Laplace transform $F(s)$ of a signal $f(t)$ can be viewed as the Fourier transform of the signal defined by $f(t)e^{-\sigma}$, i.e.,

$$F(s) = \mathcal{L}\{f(t)\} = \mathcal{F}\{f(t)e^{-\sigma}\}.$$

6.1.2 Key Properties of the Laplace Transform

Let $X(s)$ and $Y(s)$ be the Laplace transforms of $x(t)$ and $y(t)$, respectively, with corresponding regions of convergence of $R_{x-} < Re[s] < R_{x+}$ and $R_{y-} < Re[s] < R_{y+}$. Unless otherwise noted, these results generally apply to all three forms of the Laplace transform though they are defined for the bilateral transform here. Let a and b be complex constants.

Linearity.

$$\mathcal{L}\{ax(t) + by(t)\} = aX(s) + bY(s), \quad R_- < Re[s] < R_+$$

where $R_- = \max\{R^{x^-}, R^{y^-}\}$ and $R_- = \min\{R^{x^-}, R^{y^-}\}$. The region of convergence is at least as big as the intersection of the regions of convergence of $x(t)$ and $y(t)$.

Time Shift.

$$\mathcal{L}\{x(t-\tau)\} = e^{s\tau}X(s), \quad R_{x^-} < Re[s] < R_{x^+}.$$

Time Scaling (Similarity).

$$\mathcal{L}\{x(at)\} = \frac{1}{|a|}X\left(\frac{s}{a}\right), \quad R_{x^-} < |a|^{-1}Re[s] < R_{x^+}.$$

The region of convergence criterion for this exampled may be stated as, "s is in the region of convergence of $x(at)$ if s/a is in the region of convergence of $X(s)$."

Time Reversal.

$$\mathcal{L}\{x(-t)\} = X(-s), \quad -R_{x^+} < Re[s] < -R_{x^-}.$$

Note the reversal of the region of convergence.

Time Differentiation.

$$\mathcal{L}\left\{\frac{d}{dt}x(t)\right\} = sX(s), \quad R_{x^-} < Re[s] < R_{x^+}.$$

Note the introduction of a zero at the origin which may impact the region of convergence. The region of convergence is at least as big as the region of convergence of $X(s)$. For the unilateral (one-sided) zero-positive version of the Laplace transform:

$$\mathcal{L}_+\left\{\frac{d}{dt}x(t)\right\} = sX(s) - x(0^+), \quad R_{x^-} < Re[s] < R_{x^+}.$$

where $x(0^+)$ is defined as the limit of $x(t)$ as $t \longrightarrow 0$ from above. Differentiaing a second time:

$$\mathcal{L}_+\left\{\frac{d^2}{dt^2}x(t)\right\} = s^2X(s) - sx(0^+) - x'(0^+), \quad R_{x^-} < Re[s] < R_{x^+}.$$

where $x'(t) = dx(t)/dt$. In general,

$$\mathcal{L}_+\left\{\frac{d^2 = n}{dt^n}x(t)\right\} = s^nX(s) - \sum_{k=0}^{n-1} s^k x^{(k)}(0^+), \quad R_{x^-} < Re[s] < R_{x^+}.$$

Differentiation in s.

$$\mathcal{L}\{tx(t)u(t)\} = -\frac{d}{ds}X(s), \quad R_{x^-} < Re[s] < R_{x^+}$$

so that the region of convergence is unchanged. More generally,

$$\mathcal{L}\{(-t)^n x(t)u(t)\} = -\frac{d^n}{ds^n}X(s), \quad R_{x^-} < Re[s] < R_{x^+}.$$

Time Integration.
Two-sided case:

$$\mathcal{L}\left\{\int_{-\infty}^{t} x(\tau)d\tau\right\} = \frac{1}{s}X(s), \quad \max\{R_{x^-}, 0\} < Re[s] < \min\{R_{x^+}, 0\}.$$

The region of convergence is at least as large as the intersection of the region of convergence of $x(t)$ and $u(t)$ [$u(t)$ has a region of convergence of $Re[s] > 0$].

Note that this property is different for the unilateral (one-sided) zero-positive version of the Laplace transform:

$$\mathcal{L}_+\left\{\int_{-\infty}^{t} x(\tau)d\tau\right\} = f^{-1}(t) = \frac{1}{s}X(s)+\frac{1}{s}\frac{1}{x(0^+)}, \quad \max\{R_{x^-}, 0\} < Re[s] < \min\{R_{x^+}, 0\}.$$

Integrating twice yields:

$$\frac{1}{s^2}X(s) + \frac{1}{s^2}\frac{1}{x(0^+)} + \frac{1}{s}\frac{1}{x'(0^+)}, \quad \max\{R_{x^-}, 0\} < Re[s] < \min\{R_{x^+}, 0\}.$$

where $x'(t) = dx(t)/dt$.
Integration in s.

$$\frac{1}{t}x(t)u(t) = \int_{s}^{\infty} X(s)ds, \quad \max\{R_{x^-}, 0\} < Re[s] < \min\{R_{x^+}, 0\}.$$

Time Convolution.

$$\mathcal{L}\left\{x(t) * y(t)\right\} = X(s)Y(s), \quad R_- < Re[s] < R_+$$

where $R_- = \max\{R^{x^-}, R^{y^-}\}$ and $R_- = \min\{R^{x^-}, R^{y^-}\}$. The region of convergence is at least as big as the intersection of the two regions of convergence.
s**-plane Convolution.**

$$\mathcal{L}\{x(t)y(t)\} = \frac{1}{2\pi j}\int_{\sigma-j\infty}^{\sigma+j\infty} X(p)Y(s-p)dp, \quad R_{x^-} + R_{y^-} < Re[s] < R_{x^+} + R_{y^+}.$$

Autocorrelation.

$$\mathcal{L}\{x(t) * x(-t)\} = X(s)X(-s), \quad |Re[s]| < \min\{|R_{x^-}|, |R_{x^-}|\}.$$

Finite Difference.

$$\mathcal{L}\{x(t+\tau/2) - x(t-\tau/2)\} = 2\text{sinh}(s\tau/2)X(s), \quad R_{x^-} < Re[s] < R_{x^+}.$$

Initial Value Theorem. If $x(t) = 0$ for $t < 0$ and $x(t)$ has no singularities at $t = 0$, then

$$\lim_{t\to 0} x(t) = \lim_{s\to\infty} X(s).$$

Final Value Theorem. If $x(t) = 0$ for $t < 0$ and $x(t)$ has no singularities at $t = 0$, or equivalently, if $sF(s)$ is analytic for $Re[s] \geq 0$, then

$$\lim_{t\to\infty} x(t) = \lim_{s\to 0} sX(s).$$

6.2 Laplace Transform Tables

6.2.1 Bilateral (Two-Sided) Laplace Transform

Definition and Key Properties

	$x(t) = \mathcal{L}^{-1}\{X(s)\}$	$X(s) = \mathcal{L}\{x(t)\}$	Region of Convergence
1	$x(t)$	$\int_{0+}^{\infty} x(t)e^{-st}dt$	$R_{x-} < Re[s] < R_{x+}$
2	$\frac{1}{2\pi j}\int_{\sigma-j\infty}^{\sigma+j\infty} F(s)e^{st}ds$	$X(s)$	$R_{x-} < Re[s] < R_{x+}$
3	$ax(t) + by(t)$	$aX(s) + bY(s)$	$R_{-} < Re[s] < R_{+}$
		$R_{-} = \max\{R^{x^-}, R^{y^-}\}$	
		$R_{-} = \min\{R^{x^-}, R^{y^-}\}$	
4	$x(t-\tau)$	$e^{s\tau}X(s)$	$R_{x-} < Re[s] < R_{x+}$
5	$x(at)$	$\frac{1}{\lvert a\rvert}X\left(\frac{s}{a}\right)$	$R_{x-} < \lvert a\rvert^{-1}Re[s] < R_{x+}$
6	$x(-t)$	$X(-s)$	$-R_{x+} < Re[s] < -R_{x-}$
7	$\frac{d}{dt}x(t)$	$sX(s)$	$R_{x-} < Re[s] < R_{x+}$
8	$tx(t)u(t)$	$-\frac{d}{ds}X(s)$	$R_{x-} < Re[s] < R_{x+}$
9	$\int_{-\infty}^{t} x(\tau)d\tau$	$\frac{1}{s}X(s)$	$\max\{R_{x-}, 0\} < Re[s] < \min\{R_{x+}, 0\}$
10	$\frac{1}{t}x(t)u(t)$	$\int_{s}^{\infty} X(s)ds$	$\max\{R_{x-}, 0\} < Re[s] < \min\{R_{x+}, 0\}$
11	$x(t) * y(t)$	$X(s)Y(s)$	$R_{x-} < Re[s] < R_{x+}$, $R_{y-} < Re[s] < R_{y+}$
12	$x(t)y(t)$	$\frac{1}{2\pi j}\int_{\sigma-j\infty}^{\sigma+j\infty} X(p)Y(s-p)dp$	$R_{x-} + R_{y-} < Re[s] < R_{x+} + R_{y+}$
13	$x(t) * x(-t)$	$X(s)X(-s)$	$\lvert Re[s]\rvert < \min\{\lvert R_{x-}\rvert, \lvert R_{x-}\rvert\}$
14	$x(t+\tau/2) - x(t-\tau/2)$	$2\sinh(s\tau/2)X(s)$	$R_{x-} < Re[s] < R_{x+}$

Laplace Transform Limits

Assuming $x(t) = 0$ for $t < 0$ and that $x(t)$ has no singularities at the origin,

$$\lim_{t\to 0} x(t) = \lim_{s\to\infty} X(s)$$

$$\lim_{t\to\infty} x(t) = \lim_{s\to 0} sX(s)$$

Two-Sided Laplace Transform Table

Two-Sided LaPlace Transform Table

	$x(t) = \mathcal{L}^{-1}\{X(s)\}$	$X(s) = \mathcal{L}\{x(t)\}$	Region of Convergence
1	$e^{-\alpha t}u(t)$	$\frac{1}{s+\alpha}$	$Re[\alpha] < Re[s]$
2	$-e^{\alpha t}u(t)$	$\frac{1}{s+\alpha}$	$Re[s] < -Re[\alpha]$
3	$u(t)$	$\frac{1}{s}$	$0 < Re[s]$
4	$-u(-t)$	$\frac{1}{s}$	$Re[s] < 0$
5	$tu(t)$	$\frac{1}{s^2}$	$0 < Re[s]$
6	$\delta(t)$	1	all s
7	$\delta'(t)$	s	all s
8	$(1-e^{-\alpha t})u(t)$	$\frac{\alpha}{s(s+\alpha)}$	$\max\{0, -Re[\alpha]\} < Re[s]$
9	$\Pi(t)$	$\frac{\sinh \frac{1}{2}s}{\frac{1}{2}s}$	all s
10	$\Pi(t-\frac{1}{2})$	$\frac{1-e^{-s}}{s}$	all s
11	$\Lambda(t)$	$\left(\frac{\sinh \frac{1}{2}s}{\frac{1}{2}s}\right)^2$	all s
12	sgn t	$\frac{2}{s}$	$Re[s] = 0$
13	$\frac{t^n}{n!}u(t)$	$\frac{1}{s^{n+1}}$	$Re[s] > 0$
14	$-\frac{t^n}{n!}u(-t)$	$\frac{1}{s^{n+1}}$	$Re[s] < 0$
15	$t^2 e^{-\alpha t}u(t)$	$\frac{2}{(s+\alpha)^3}$	$Re[s] > -\alpha$
16	$\frac{t^n}{n!}e^{-\alpha t}u(t)$	$\frac{1}{(s+\alpha)^{n+1}}$	$Re[s] > -\alpha$
17	$-\frac{t^n}{n!}e^{-\alpha t}u(-t)$	$\frac{1}{(s+\alpha)^{n+1}}$	$Re[s] < -\alpha$
18	$[\cos \alpha t]u(t)$	$\frac{s}{s^2+\alpha^2}$	$Re[s] > 0$
19	$[\sin \alpha t]u(t)$	$\frac{\alpha}{s^2+\alpha^2}$	$Re[s] > 0$
20	$[e^{-\beta t}\cos \alpha t]u(t)$	$\frac{(s+\beta)}{(s+\beta)^2+\alpha^2}$	$Re[s] > -\beta$
21	$[e^{-\beta t}\sin \alpha t]u(t)$	$\frac{\alpha}{(s+\beta)^2+\alpha^2}$	$Re[s] > -\beta$

Unilateral Laplace Transform Transform Table†

	$X(s) = \mathcal{L}_+\{x(t)\}$	$x(t) = \mathcal{L}_+^{-1}\{X(s)\}$
1	$\frac{1}{s}$	$1^\dagger$
2	1	$\delta(t)$
3	s	$\delta'(t)$
4	$\frac{1}{s+a}$	e^{-at}
5	$\frac{1}{s^2}$	t
6	$\frac{1}{(s+a)(s+c)}$	$\frac{e^{-at}-e^{-ct}}{c-a}$
7	$\frac{s+a_0}{(s+a)(s+c)}$	$\frac{(a_0-a)e^{-at}-(a_0-c)e^{-ct}}{c-a}$
8	$\frac{1}{s(s+a)(s+c)}$	$\frac{1}{ac}+\frac{ce^{-at}-ae^{-ct}}{ac(a-c)}$
9	$\frac{s+a_0}{s(s+a)(s+c)}$	$\frac{a_0}{ac}+\frac{a_0-a}{a(a-c)}e^{-at}+\frac{a_0-c}{c(c-a)}e^{-ct}$
10	$\frac{1-e^{-s}}{s}$	$\Pi(t-\frac{1}{2})$
11	$\frac{1}{(s+\alpha)^{n+1}}$	$\frac{t^n}{n!}e^{-\alpha t}$
12	$\frac{s}{s^2+\alpha^2}$	$\cos\alpha t$
13	$\frac{\alpha}{s^2+\alpha^2}$	$\sin\alpha t$
14	$\frac{(s+\beta)}{(s+\beta)^2+\alpha^2}$	$e^{-\beta t}\cos\alpha t$
15	$\frac{\alpha}{(s+\beta)^2+\alpha^2}$	$e^{-\beta t}\sin\alpha t$

† Note that an implied $u(t)$ is associated with all time domin signals.

Unilateral Laplace Transform Transform Table [Continued]

	$X(s) = \mathcal{L}_+\{x(t)\}$	$x(t) = \mathcal{L}_+^{-1}\{X(s)\}$
21	$\dfrac{s^2 + a_1 s + a_0}{s(s+a)(s+c)}$	$\dfrac{a_0}{ac} + \dfrac{a^2 - a_1 a + a_0}{a(a-c)} e^{-at} + \dfrac{c^2 - a_1 c + a_0}{c(c-a)} e^{-ct}$
22	$\dfrac{1}{(s+a)(s+c)(s+d)}$	$\dfrac{e^{-at}}{(c-a)(d-a)} + \dfrac{e^{-ct}}{(a-c)(d-c)} + \dfrac{e^{-dt}}{(a-d)(c-d)}$
23	$\dfrac{s + a_0}{(s+a)(s+c)(s+d)}$	$\dfrac{a_0 - a}{(c-a)(d-a)} e^{-at} + \dfrac{a_0 - c}{(a-c)(d-c)} e^{-ct} + \dfrac{a_0 - d}{(a-d)(c-d)} e^{-dt}$
24	$\dfrac{s^2 + a_1 s + a_0}{(s+a)(s+c)(s+d)}$	$\dfrac{a^2 - a_1 a + a_0}{(c-a)(d-a)} e^{-at} + \dfrac{c^2 - a_1 c + a_0}{(a-c)(d-c)} e^{-ct} + \dfrac{d^2 - a_1 d + a_0}{(a-d)(c-d)} e^{-dt}$
25	$\dfrac{1}{s^2 + b^2}$	$\dfrac{1}{b} \sin bt$
26	$\dfrac{1}{s^2 - b^2}$	$\dfrac{1}{b} \sinh bt$
27	$\dfrac{s}{s^2 + b^2}$	$\cos bt$
28	$\dfrac{s}{s^2 - b^2}$	$\cosh bt$
29	$\dfrac{s + a_0}{s^2 + b^2}$	$\dfrac{1}{b}(a_0^2 + b^2)^{\frac{1}{2}} \sin(bt + \psi)$, $\psi \triangleq \tan^{-1} \dfrac{b}{a_0}$
30	$\dfrac{1}{s(s^2 + b^2)}$	$\dfrac{1}{b^2}(1 - \cos bt)$
31	$\dfrac{s + a_0}{s(s^2 + b^2)}$	$\dfrac{a_0}{b^2} - \dfrac{(a_0^2 + b^2)^{\frac{1}{2}}}{b^2} \cos(bt + \psi)$, $\psi \triangleq \tan^{-1} \dfrac{b}{a_0}$
32	$\dfrac{s^2 + a_1 s + a_0}{s(s^2 + b^2)}$	$\dfrac{a_0}{b^2} - \dfrac{[(a_0 + b^2)^2 + a_1^2 b^2]^{\frac{1}{2}}}{b^2} \cos(bt + \psi)$, $\psi \triangleq \tan^{-1} \dfrac{a_1 b}{a_0 - b^2}$
33	$\dfrac{s + a_0}{(s+a)(s^2 + b^2)}$	$\dfrac{a_0 - a}{a^2 + b^2} e^{-at} + \dfrac{1}{b}\left[\dfrac{a_0^2 + b^2}{a^2 + b^2}\right]^{\frac{1}{2}} \sin(bt + \psi)$, $\psi \triangleq \tan^{-1} \dfrac{b}{a_0} - \tan^{-1} \dfrac{b}{a}$

Unilateral Laplace Transform Transform Table [Continued]

	$X(s) = \mathcal{L}_+\{x(t)\}$	$x(t) = \mathcal{L}_+^{-1}\{X(s)\}$
34	$\dfrac{s^2 + a_1 s + a_0}{(s+a)(s^2+b^2)}$	$\dfrac{a^2 - a_1 a + a_0}{a^2+b^2}e^{-at} + \dfrac{1}{b}\left[\dfrac{(a_0-b^2)^2 + a_1^2 b^2}{a^2+b^2}\right]^{\frac{1}{2}} \sin(bt+\psi)$ $\psi \triangleq \tan^{-1}\dfrac{a_1 b}{a_0 - b^2} - \tan^{-1}\dfrac{b}{a}$
35	$\dfrac{s + a_0}{s(s+a)(s^2+b^2)}$	$\dfrac{a_0}{ab^2} + \dfrac{a - a_0}{a(a^2+b^2)}e^{-at} - \dfrac{1}{b^2}\left[\dfrac{a_0^2+b^2}{a^2+b^2}\right]^{\frac{1}{2}} \cos(bt+\psi)$ $\psi \triangleq \tan^{-1}\dfrac{b}{a_0} - \tan^{-1}\dfrac{b}{a}$
36	$\dfrac{s + a_1 s + a_0}{s(s+a)(s^2+b^2)}$	$\dfrac{a_0}{ab^2} - \dfrac{a^2 - a_1 a + a_0}{a(a^2+b^2)}e^{-at}$ $-\dfrac{1}{b^2}\left[\dfrac{(a_0-b^2)^2 + a_1^2 b^2}{a^2+b^2}\right]^{\frac{1}{2}} \cos(bt+\psi)$ $\psi \triangleq \tan^{-1}\dfrac{a_1 b}{a_0 - b^2} - \tan^{-1}\dfrac{b}{a}$
37	$\dfrac{s^2 + a_1 s + a_0}{(s+a)(s+c)(s^2+b^2)}$	$\dfrac{a^2 - a_1 a + a_0}{(c-a)(a^2+b^2)}e^{-at} + \dfrac{c^2 - a_1 c + a_0}{(a-c)(c^2+b^2)}e^{-ct}$ $+\dfrac{1}{b}\left[\dfrac{(a_0-b^2)^2 + a_1^2 b^2}{(a^2+b^2)(c^2+b^2)}\right]^{\frac{1}{2}} \sin(bt+\psi)$ $\psi \triangleq \tan^{-1}\dfrac{a_1 b}{a_0 - b^2} - \tan^{-1}\dfrac{b}{a} - \tan^{-1}\dfrac{b}{c}$
38	$\dfrac{s^2 + a_2 s^2 + a_1 s + a_0}{(s+a)(s+c)(s^2+b^2)}$	$\dfrac{-a^3 + a_2 a^2 - a_1 a + a_0}{(c-a)(a^2+b^2)}e^{-at} + \dfrac{-c^3 + a_2 c^2 - a_1 c + a_0}{(a-c)(c^2+b^2)}e^{-ct}$ $+\dfrac{1}{b}\left[\dfrac{(a_0 - a_2 b^2)^2 + b^2(a_1 - b^2)^2}{(a^2+b^2)(c^2+b^2)}\right]^{\frac{1}{2}} \sin(bt+\psi)$ $\psi \triangleq \tan^{-1}\dfrac{b(a_1 - b^2}{a_0 - a_2 b^2} - \tan^{-1}\dfrac{b}{a} - \tan^{-1}\dfrac{b}{c}$
39	$\dfrac{s}{(s^2+b^2)(s^2+c^2)}$	$\dfrac{\cos bt - \cos ct}{c^2 - b^2}$
40	$\dfrac{s}{[s^2+(b+c)^2][s^2+(b-c)^2]}$	$\dfrac{1}{2cb}\sin ct \times \sin bt$
41	$\dfrac{s^2 + a_1 s + a_0}{(s^2+b^2)(s^2+c^2)}$	$\dfrac{[(a_0-b^2)^2 + a_1^2 b^2]^{\frac{1}{2}}}{b(c^2-b^2)} \sin(bt+\psi_1)$ $+\dfrac{[(a_0+c^2)^2 + a_1^2 c^2]^{\frac{1}{2}}}{c(b^2-c^2)} \sin(ct+\psi_2)$ $\psi_1 \triangleq \tan^{-1}\dfrac{a_1 b}{a_0 - b^2}; \psi_2 \triangleq \tan^{-1}\dfrac{a_1 c}{a_0 - c^2}$

Unilateral Laplace Transform Transform Table [Continued]

	$X(s) = \mathcal{L}_+\{x(t)\}$	$x(t) = \mathcal{L}_+^{-1}\{X(s)\}$
42	$\frac{s^3 + a_2 s^2 + a_1 s + a_0}{(s^2+b^2)(s^2+c^2)}$	$\frac{[(a_0 - a_2 b^2)^2 + b^2(a_1 - b^2)^2]^{\frac{1}{2}}}{b(c^2-b^2)} \sin(bt + \psi_1)$ $+\frac{[(a_0 + a_2 c^2)^2 + c^2(a_1 - c^2)^2]^{\frac{1}{2}}}{c(b^2-c^2)} \sin(ct + \psi_2)$ $\psi_1 \triangleq \tan^{-1} \frac{b(a_1 - b^2)}{a_0 - a_2 b^2}; \psi_2 \triangleq \tan^{-1} \frac{c(a_1 - c^2)}{a_0 - a_2 c^2}$
43	$\frac{1}{(s+a)^2+b^2}$	$\frac{1}{b} e^{-at} \sin bt$
44	$\frac{s + a_0}{(s+a)^2+b^2}$	$\frac{1}{b}[(a_0 - a)^2 + b^2]^{\frac{1}{2}} e^{-at} \sin(bt + \psi)$ $\psi \triangleq \tan^{-1} \frac{b}{a_0 - a}$
45	$\frac{s+a}{(s+a)^2+b^2}$	$e^{-at} \cos bt$
46	$\frac{1}{s[(s+a)^2+b^2]}$	$\frac{1}{b_0^2} + \frac{1}{b_0 b} e^{-at} \sin(bt - \psi)$ $\psi \triangleq \tan^{-1} \frac{b}{-a}$ $b_0^2 \triangleq a^2 + b^2$
47	$\frac{s + a_0}{s[(s+a)^2+b^2]}$	$\frac{a_0}{b_0^2} + \frac{1}{b b_0}[(a_0 - a)^2 + b^2]^{\frac{1}{2}} e^{-at} \sin(bt + \psi)$ $\psi \triangleq \tan^{-1} \frac{b}{a_0 - a} - \tan^{-1} \frac{b}{-a}$ $b_0^2 \triangleq a^2 + b^2$
48	$\frac{s^2 + a_1 s + a_0}{s[(s+a)^2+b^2]}$	$\frac{a_0}{b_0^2} + \frac{1}{b b_0}[(a^2 - b^2 - a_1 a + a_0)^2$ $+b^2(a_1 - 2a)^2]^{\frac{1}{2}} e^{-at} \sin(bt + \psi)$ $\psi \triangleq \tan^{-1} \frac{b(a_1 - 2a)}{a^2 - b^2 - a_1 a + a_0} - \tan^{-1} \frac{b}{-a}$ $b_0^2 \triangleq a^2 + b^2$
49	$\frac{1}{(s+c)[(s+a)^2+b^2]}$	$\frac{e^{-ct}}{(c-a)^2+b^2} + \frac{1}{b[(c-a)^2+b^2]^{\frac{1}{2}}} e^{-at} \sin(bt - \psi)$ $\psi \triangleq \tan^{-1} \frac{b}{c-a}$
50	$\frac{s + a_0}{(s+c)[(s+a)^2+b^2]}$	$\frac{a_0 - c}{(a-c)^2+b^2} e^{-ct}$ $+\frac{1}{b}\left[\frac{(a_0 - a)^2 + b^2}{(c-a)^2+b^2}\right]^{\frac{1}{2}} e^{-at} \sin(bt + \psi)$ $\psi \triangleq \tan^{-1} \frac{b}{a_0 - a} - \tan^{-1} \frac{b}{c-a}$

Unilateral Laplace Transform Transform Table [Continued]

	$X(s) = \mathcal{L}_+\{x(t)\}$	$x(t) = \mathcal{L}_+^{-1}\{X(s)\}$
51	$\frac{s^2 + a_1 s + a_0}{(s+c)[(s+a)^2 + b^2]}$	$\frac{c^2 - a_1 c + a_0}{(a-c)^2 + b^2} e^{-ct} + \frac{1}{b}\left[\frac{(a^2 - b^2 - a_1 a + a_0)^2 + b^2(a_1 - 2a)^2}{(c-a)^2 + b^2}\right]^{\frac{1}{2}} e^{-at}\sin(bt + \psi)$ $\psi \triangleq \tan^{-1}\frac{b(a_1 - 2a)}{a^2 - b^2 - a_1 a + a_0} - \tan^{-1}\frac{b}{c-a}$
52	$\frac{1}{s(s+c)[(s+a)^2 + b^2]}$	$\frac{1}{cb_0^2} - \frac{1}{c[(a-c)^2 + b^2]} e^{-ct} + \frac{1}{bb_0[(c-a)^2 + b^2]^{\frac{1}{2}}} e^{-at}\sin(bt - \psi)$ $\psi \triangleq \tan^{-1}\frac{b}{-a} - \tan^{-1}\frac{b}{c-a}$ $b_0^2 \triangleq a^2 + b^2$
53	$\frac{s + a_0}{s(s+c)[(s+a)^2 + b^2]}$	$\frac{a_0}{cb_0^2} + \frac{c - a_0}{c[(a-c)^2 + b^2]} e^{-ct} + \frac{1}{bb_0}\left[\frac{(a_0 - a)^2 + b^2}{(c-a)^2 + b^2}\right]^{\frac{1}{2}} e^{-at}\sin(bt + \psi)$ $\psi \triangleq \tan^{-1}\frac{b}{a_0 - a} - \tan^{-1}\frac{b}{c-a} - \tan^{-1}\frac{b}{-a}$ $b_0^2 \triangleq a^2 + b^2$
54	$\frac{s^2 + a_1 s + a_0}{(s+c)(s+d)[(s+a)^2 + b^2]}$	$\frac{c^2 - a_1 c + a_0}{(d-c)[(a-c)^2 + b^2]} e^{-ct} + \frac{d^2 - a_1 d + a_0}{(c-d)[(a-d)^2 + b^2]} e^{-dt} + \frac{1}{b}\left\{\frac{(a^2 - b^2 - a_1 a + a_0)^2 + b^2(a_1 - 2a)^2}{[(d-a)^2 + b^2][(c-a)^2 + b^2]}\right\}^{\frac{1}{2}} e^{-at}\sin(bt + \psi)$ $\psi \triangleq \tan^{-1}\frac{b(a_1 - 2a)}{a^2 - b^2 - a_1 a + a_0} - \tan^{-1}\frac{b}{c-a} - \tan^{-1}\frac{b}{d-a}$
55	$\frac{1}{(s^2 + \lambda^2)[(s+a)^2 + b^2]}$	$\frac{1}{[(b_0^2 - \lambda^2)^2 + 4a^2\lambda^2]^{\frac{1}{2}}}\left[\frac{1}{\lambda}\sin(\lambda t - \psi_1) + \frac{1}{b} e^{-at}\sin(bt - \psi_2)\right]$ $\psi_1 \triangleq \tan^{-1}\frac{2a\lambda}{b_0^2 - \lambda^2}; \psi_2 \triangleq \frac{-2ab}{a^2 - b^2 + \lambda^2}; b_0^2 \triangleq a^2 + b^2$
56	$\frac{s + a_0}{(s^2 + \lambda^2)[(s+a)^2 + b^2]}$	$\frac{1}{\lambda}\left[\frac{a_0^2 + \lambda^2}{(b_0^2 - \lambda^2)^2 + 4a^2\lambda^2}\right]^{\frac{1}{2}}\sin(\lambda t + \psi_1) + \frac{1}{b}\left[\frac{(a_0 - a)^2 + b^2}{(b_0^2 - \lambda^2)^2 + 4a^2\lambda^2}\right]^{\frac{1}{2}} e^{-at}\sin(bt - \psi_2)$ $\psi_1 \triangleq \tan^{-1}\frac{\lambda}{a_0} - \tan^{-1}\frac{2a\lambda}{b_0^2 - \lambda^2}$ $\psi_2 \triangleq \tan^{-1}\frac{b}{a_0 - a} - \tan^{-1}\frac{-2ab}{a^2 - b^2 + \lambda^2}$ $b_0^2 \triangleq a^2 + b^2$

Unilateral Laplace Transform Transform Table [Continued]

	$X(s) = \mathcal{L}_+\{x(t)\}$	$x(t) = \mathcal{L}_+^{-1}\{X(s)\}$
57	$\frac{s^2 + a_1 s + a_0}{(s^2+\lambda^2)[(s+a)^2+b^2]}$	$\frac{1}{\lambda}\left[\frac{(a_0+\lambda^2)^2 + a_1^2\lambda^2}{(b_0^2-\lambda^2)^2+4a^2\lambda^2}\right]^{\frac{1}{2}} \sin(\lambda t + \psi_1)$ $+\frac{1}{b}\left[\frac{(a^2-b^2-a_1 a + a_0)^2 + b^2(a_1-2a)^2}{(b_0^2-\lambda^2)^2+4a^2\lambda^2}\right]^{\frac{1}{2}} \times$ $e^{-at}\sin(bt-\psi_2)$ $\psi_1 \triangleq \tan^{-1}\frac{a_1\lambda}{a_0-\lambda^2} - \tan^{-1}\frac{2a\lambda}{b_0^2-\lambda^2}$ $\psi_2 \triangleq \tan^{-1}\frac{b(a_1-2a)}{a^2-b^2-a_1 a+a_0}$ $-\tan^{-1}\frac{-2ab}{a^2-b^2+\lambda^2}$ $b_0^2 \triangleq a^2+b^2$
58	$\frac{s+a_0}{(s+c)(s^2+\lambda^2)[(s+a)^2+b^2]}$	$\frac{a_0-c}{(\lambda^2+c^2)[(a-c)^2+b^2]}e^{-ct}+$ $\frac{1}{\lambda}\left\{\frac{a_0^2+\lambda^2}{(c^2+\lambda^2)[(b_0^2-\lambda^2)^2+4a^2\lambda^2]}\right\}^{\frac{1}{2}}\sin(\lambda t+\psi_1)$ $+\frac{1}{b}\left\{\frac{a_0-a)^2+b^2}{[(c-a)^2+b^2][(b_0^2-\lambda^2)^2+4a^2\lambda^2]}\right\}^{\frac{1}{2}} \times$ $e^{-at}\sin(bt+\psi_2)$ $\psi_1 \triangleq \tan^{-1}\frac{\lambda}{a_0} - \tan^{-1}\frac{\lambda}{c} - \tan^{-1}\frac{2a\lambda}{b_0^2-\lambda^2}$ $\psi_2 \triangleq \tan^{-1}\frac{b}{a_0-a} - \tan^{-1}\frac{b}{c-a}$ $-\tan^{-1}\frac{-2ab}{a^2-b^2+\lambda^2}$ $b_0^2 \triangleq a^2+b^2$
59	$\frac{1}{s^2}$	t
60	$\frac{1}{s^n}$	$\frac{1}{(n-1)!}t^{n-1}$ $n>0$ integer
61	$\frac{1}{(s+a)s^2}$	$\frac{e^{-at}+at-1}{a^2}$
62	$\frac{s+a_0}{(s+a)s^2}$	$\frac{a_0-a}{a^2}e^{-at} + \frac{a_0}{a}t + \frac{a-a_0}{a^2}$
63	$\frac{s^2+a_1 s+a_0}{(s+a)s^2}$	$\frac{a^2-a_1 a+a_0}{a^2}e^{-at} + \frac{a_0}{a}t + \frac{a_1 a-a_0}{a^2}$
64	$\frac{1}{(s+a)^2}$	te^{-at}
65	$\frac{s+a_0}{(s+a)^2}$	$[(a_0-a)t+1]e^{-at}$
66	$\frac{1}{(s+a)^n}$	$\frac{1}{(n-1)!}t^{n-1}e^{-at}$ $n>0$ integer
67	$\frac{s^n}{(s+a)^{n+1}}$	$e^{-at}\sum_{k=0}^{n}\frac{n!(-a)^k}{(n-k)!(k!)^2}t^k$ $n \geq 0$ integer

Unilateral Laplace Transform Transform Table [Continued]

	$X(s)=\mathcal{L}_+\{x(t)\}$	$x(t)=\mathcal{L}_+^{-1}\{X(s)\}$
68	$\frac{1}{s(s+a)^2}$	$\frac{1-(1+at)e^{-at}}{a^2}$
69	$\frac{s+a_0}{s(s+a)^2}$	$\frac{a_0}{a^2}+\left(\frac{a-a_0}{a}t-\frac{a_0}{a^2}\right)e^{-at}$
70	$\frac{s^2+a_1s+a_0}{s(s+a)^2}$	$\frac{a_0}{a^2}+\left(\frac{a_1a-a_0-a^2}{a}t+\frac{a^2-a_0}{a^2}\right)e^{-at}$
71	$\frac{1}{(s+c)(s+a)^2}$	$\frac{1}{(c-a)^2}e^{-ct}+\frac{(c-a)t-1}{(c-a)^2}e^{-at}$
72	$\frac{s+a_0}{(s+c)(s+a)^2}$	$\frac{a_0-c}{(a-c)^2}e^{-ct}+\left[\frac{a_0-a}{c-a}t+\frac{c-a_0}{(c-a)^2}\right]e^{-at}$
73	$\frac{s^2+a_1s+a_0}{(s+c)(s+a)^2}$	$\frac{c^2-a_1c+a_0}{(a-c)^2}e^{-ct}$ $+\left[\frac{a^2-a_1a+a_0}{c-a}t+\frac{a^2-2ac+a_1c-a_0}{(c-a)^2}\right]e^{-at}$
74	$\frac{s+a_0}{(s+c)(s+a)^3}$	$\frac{a_0-c}{(a-c)^3}e^{-ct}+\left[\frac{a_0-a}{2(c-a)}t^2+\frac{c-a_0}{(c-a)^2}t+\frac{a_0-c}{(c-a)^3}\right]e^{-at}$
75	$\frac{s+a_0}{s(s+c)(s+a)^2}$	$\frac{a_0}{ca^2}+\frac{c-a_0}{c(a-c)^2}e^{-ct}+\left[\frac{a_0-a}{a(a-c)}t+\frac{2a_0a-a^2-a_0c}{a^2(a-c)^2}\right]e^{-at}$
76	$\frac{s^2+a_1s+a_0}{s(s+c)(s+a)^2}$	$\frac{a_0}{ca^2}-\frac{c^2-a_1c+a_0}{c(a-c)^2}e^{-ct}$ $+\left[\frac{a^2-a_1a+a_0}{a(a-c)}t+\frac{(c-a_1)a^2+(2a-c)a_0}{a^2(a-c)^2}\right]e^{-at}$
77	$\frac{s+a_0}{(s+d)(s+c)(s+a)^2}$	$\frac{a_0-c}{(d-c)(a-c)^2}e^{-ct}+\frac{a_0-d}{(c-d)(a-d)^2}e^{-dt}$ $+\left[\frac{a_0-a}{(c-a)(d-a)}t+\frac{2a_0a-a^2-a_0(c+d)+cd}{(c-a)^2(d-a)^2}\right]e^{-at}$
78	$\frac{s+a_0}{(s+a)(s+c)s^2}$	$\frac{a_0-a}{a^2(c-a)}e^{-at}+\frac{a_0-c}{c^2(a-c)}e^{-ct}+\frac{a_0}{ac}t+\frac{ac-a_0(a+c)}{a^2c^2}$
79	$\frac{s^2+a_1s+a_0}{(s+a)(s+c)s^2}$	$\frac{a^2-a_1a+a_0}{a^2(c-a)}e^{-at}+\frac{c^2-a_1c+a_0}{c^2(a-c)}e^{-ct}$ $+\frac{a_0}{ac}t+\frac{a_1ac-a_0(a+c)}{a^2c^2}$
80	$\frac{s^2+a_1s+a_0}{(s+a)^2s^2}$	$\left[\frac{a^2-a_1a+a_0}{a^2}t+\frac{2a_0-a_1a}{a^3}\right]e^{-at}+\frac{a_0}{a^2}t+\frac{a_1a-2a_0}{a^3}$
81	$\frac{s+a_0}{(s+a)^2(s+c)^2}$	$\left[\frac{a_0-a}{(c-a)^2}t+\frac{a+c-2a_0}{(c-a)^3}\right]e^{-at}$ $+\left[\frac{a_0-c}{(a-c)^2}t+\frac{a+c-2a_0}{(a-c)^3}\right]e^{-ct}$

Unilateral Laplace Transform Transform Table [Continued]

	$X(s) = \mathcal{L}_+\{x(t)\}$	$x(t) = \mathcal{L}_+^{-1}\{X(s)\}$
82	$\frac{s^2+a_1s+a_0}{(s+a)^2(s+c)^2}$	$\left[\frac{a^2-a_1a+a_0}{(c-a)^2}t+\frac{a_1(a+c)-2(ac+a_0)}{(c-a)^3}\right]e^{-at} + \left[\frac{c^2-a_1c+a_0}{(c-a)^2}t-\frac{a_1(a+c)-2(ac+a_0)}{(c-a)^3}\right]e^{-ct}$
83	$\frac{s^2+a_1s+a_0}{(s+a)^3s^2}$	$\left(\frac{a^2-a_1a+a_0}{2a^2}t^2+\frac{2a_0-a_1a}{a^3}t+\frac{3a_0-a_1a}{a^4}\right)e^{-at} + \frac{a_0}{a^3}t+\frac{a_1a-3a_0}{a^4}$
84	$\frac{1}{(s^2+b^2)s^2}$	$\frac{1}{b^2}t-\frac{1}{b^3}\sin bt$
85	$\frac{1}{(s^2-b^2)s^2}$	$\frac{1}{b^3}\sinh bt-\frac{1}{b^2}t$
86	$\frac{s+a_0}{(s^2+b^2)s^2}$	$\frac{a_0}{b^2}t+\frac{1}{b^2}-\frac{1}{b^3}(a_0+b^2)^{\frac{1}{2}}\sin(bt+\psi)$ $\psi \triangleq \tan^{-1}\frac{b}{a_0}$
87	$\frac{s^2+a_1s+a_0}{(s^2+b^2)s^2}$	$\frac{a_0}{b^2}t+\frac{a_1}{b^2}-\frac{1}{b^3}[(a_0^2+b^2)^2+a_1^2b^2]^{\frac{1}{2}}\sin(bt+\psi)$ $\psi \triangleq \tan^{-1}\frac{a_1b}{a_0-b^2}$
88	$\frac{1}{(s^2+b^2)s^3}$	$\frac{1}{b^4}(\cos bt-1)+\frac{1}{2b^2}t^2$
89	$\frac{1}{(s^2-b^2)s^3}$	$\frac{1}{b^4}(\cosh bt-1)-\frac{1}{2b^2}t^2$
90	$\frac{1}{(s^2+b^2)(s+a)^2}$	$\frac{1}{b(a^2+b^2)}\sin(bt-\psi)+\left[\frac{1}{a^2+b^2}t+\frac{2a}{(a^2+b^2)^2}\right]e^{-at}$ $\psi \triangleq 2\tan^{-1}\frac{b}{a}$
91	$\frac{s+a_0}{(s^2+b^2)(s+a)^2}$	$\frac{(a_0^2+b^2)^{\frac{1}{2}}}{b(a^2+b^2)}\sin(bt+\psi)+\left[\frac{a_0-a}{a^2+b^2}t+\frac{2a_0a+b^2-a^2}{(a^2+b^2)^2}\right]e^{-at}$ $\psi \triangleq \tan^{-1}\frac{b}{a_0}-2\tan^{-1}\frac{b}{a}$
92	$\frac{s^2+a_1s+a_0}{(s^2+b^2)(s+a)^2}$	$\frac{[(a_0-b^2)^2+a_1^2b^2]^{\frac{1}{2}}}{b(a^2+b^2)}\sin(bt+\psi) + \left[\frac{a^2-a_1a+a_0}{a^2+b^2}t+\frac{a_1(b^2-a^2)+2a(a_0-b^2)}{(a^2+b^2)^2}\right]e^{-at}$ $\psi \triangleq \tan^{-1}\frac{a_1b}{a_0-b^2}-2\tan^{-1}\frac{b}{a}$

Unilateral Laplace Transform Transform Table [Continued]

	$X(s)=\mathcal{L}_+\{x(t)\}$	$x(t)=\mathcal{L}_+^{-1}\{X(s)\}$
93	$\frac{s+a_0}{s(s^2+b^2)(s+a)^2}$	$\frac{a_0}{b^2a^2}-\frac{(a_0^2+b^2)^{\frac{1}{2}}}{b^2(a^2+b^2)}\cos(bt+\psi)$ $+\left[\frac{a-a_0}{a(a^2+b^2)}t+\frac{2a^3-3a_0a^2-a_0b^2}{a^2(a^2+b^2)^2}\right]e^{-at}$ $\psi\triangleq\tan^{-1}\frac{b}{a_0}-2\tan^{-1}\frac{b}{a}$
94	$\frac{s^2+a_1s+a_0}{(s^2+b^2)(s+a)^2}$	$\frac{c^2-a_1c+a_0}{(c^2+b^2)(a-c)^2}e^{-ct}+\frac{[(a_0-b^2)^2+a_1^2b^2]^{\frac{1}{2}}}{b(c^2+b^2)^{\frac{1}{2}}(a^2+b^2)}\sin(bt+\psi)+$ $\frac{a^2-a_1a+a_0}{(c-a)(a^2+b^2)}te^{-at}$ $+\frac{(c-a)(a^2+b^2)(a_1-2a)-(a^2-a_1a+a_0)(3a^2+b^2-2ac)}{(c-a)^2(a^2+b^2)^2}e^{-at}$ $\psi\triangleq\tan^{-1}\frac{a_1b}{a_0-b^2}-\tan^{-1}\frac{b}{c}-2\tan^{-1}\frac{b}{a}$
95	$\frac{1}{(s^2+b^2)^2}$	$\frac{1}{2b^3}(\sin bt-bt\cos bt)$
96	$\frac{s}{(s^2+b^2)^2}$	$\frac{1}{2b}t\sin bt$
97	$\frac{s^2}{(s^2+b^2)^2}$	$\frac{1}{2b}(\sin bt+bt\cos bt)$
98	$\frac{s^2-b^2}{(s^2+b^2)^2}$	$t\cos bt$
99	$\frac{1}{s(s^2+b^2)^2}$	$\frac{1}{b^4}(1-\cos bt)-\frac{1}{2b^3}t\sin bt$
100	$\frac{s^2+a_1s+a_0}{s(s^2+b^2)^2}$	$\frac{a_0}{b^4}-\frac{[(a_0-b^2)^2+a_1^2b^2]^{\frac{1}{2}}}{2b^3}t\sin(bt+\psi_1)$ $-\frac{(4a_0^2+a_1^2b^2)^{\frac{1}{2}}}{2b^4}\cos(bt+\psi_2)$ $\psi_1\triangleq\tan^{-1}\frac{a_1b}{a_0-b^2}$; $\psi_2\triangleq\tan^{-1}\frac{a_1b}{2a_0}$
101	$\frac{1}{[(s+a)^2+b^2]s^2}$	$\frac{1}{b_0^2}\left[t-\frac{2a}{b_0^2}+\frac{1}{b}e^{-at}\sin(bt-\psi)\right]$ $\psi\triangleq 2\tan^{-1}\frac{b}{-a}$; $b_0^2\triangleq a^2+b^2$
102	$\frac{1}{(s+c)^2[(s+a)^2+b^2]}$	$\frac{1}{(a-c)^2+b^2}\left[te^{-ct}+\frac{2(c-a)}{(a-c)^2+b^2}e^{-at}+\frac{1}{b}e^{-at}\sin(bt-\psi)\right]$ $\psi\triangleq 2\tan^{-1}\frac{b}{c-a}$

Unilateral Laplace Transform Transform Table [Continued]

	$X(s) = \mathcal{L}_+\{x(t)\}$	$x(t) = \mathcal{L}_+^{-1}\{X(s)\}$
103	$\dfrac{s^2 + a_1 s + a_0}{(s+c)^2[(s+a)^2+b^2]}$	$\dfrac{c^2 - a_1 c + a_0}{(a-c)^2+b^2} te^{-ct} +$ $\dfrac{[(a-c)^2+b^2](a_1-2c) - 2(a-c)(c^2-a_1c+a_0)}{[(a-c)^2+b^2]^2} e^{-at} +$ $+\dfrac{[(a^2-b^2-a_1a+a_0)^2 + b^2(a_1-2a)^2]^{\frac{1}{2}}}{b[(c-a)^2+b^2]} e^{-at}\sin(bt+\psi)$ $\psi \triangleq \tan^{-1}\dfrac{b(a_1-2a)}{a^2-b^2-a_1a+a_0} - 2\tan^{-1}\dfrac{b}{c-a}$
104	$\dfrac{1}{[(s+a)^2+b^2]^2}$	$\dfrac{1}{2b^3} e^{-at}(\sin bt - bt\cos bt)$
105	$\dfrac{s+a}{[(s+a)^2+b^2]^2}$	$\dfrac{1}{2b^3} te^{-at}\sin bt$
106	$\dfrac{s^2+a_0}{[(s+a)^2+b^2]^2}$	$\dfrac{b_0^2+a_0}{2b^3} e^{-at}\sin bt - \dfrac{[(a^2-b^2+a_0)^2+4a^2b^2]^{\frac{1}{2}}}{2b^2} te^{-at}\cos(bt+\psi)$ $\psi \triangleq \tan^{-1}\dfrac{-2ab}{a^2-b^2+a_0}$; $b_0^2 \triangleq a^2+b^2$
107	$\dfrac{(s+a)^2-b^2}{[(s+a)^2+b^2]^2}$	$te^{-at}\cos bt$
108	$\tan^{-1}\dfrac{b}{s}$	$\dfrac{\sin bt}{t}$
109	$\ln\dfrac{s+b}{s+a}$	$\dfrac{e^{-at}-e^{-bt}}{t}$
110	$e^{\frac{s^2}{4a}}$ cerf $\frac{s}{2\sqrt{a}}$	$2\sqrt{\dfrac{a}{\pi}} e^{-at^2}$ cerf $y \triangleq 1-$erf $y \triangleq 1 - \frac{2}{\sqrt{\pi}}\int_0^y e^{-x^2}dx$
111	$\dfrac{1}{\sqrt{s^2+a^2}}$	$J_0(at)$
112	$\dfrac{1}{\sqrt{s^2+a^2}(\sqrt{s^2+a^2}+s)}$	$\dfrac{1}{a}J_1(at)$
113	$\dfrac{1}{\sqrt{s^2+a^2}(\sqrt{s^2+a^2}+s)^n}$	$\dfrac{1}{a^n}J_n(at)$ $n \geq 0$ integer
114	$\dfrac{1}{\sqrt{s^2+a^2}(\sqrt{s^2+a^2}+s)^n}$	$\dfrac{1}{a^n}\displaystyle\int_0^t J_n(at)dt$ $n \geq 0$ integer
115	$\dfrac{1}{\sqrt{s^2+a^2}+s}$	$\dfrac{1}{a}\dfrac{J_1(at)}{t}$
116	$\dfrac{1}{(\sqrt{s^2+a^2}+s)^n}$	$\dfrac{n}{a^n}\dfrac{J_n(at)}{t}$ $n > 0$ integer
117	$\dfrac{1}{s(\sqrt{s^2+a^2}+s)^n}$	$\dfrac{n}{a^n}\displaystyle\int_0^t \dfrac{J_n(at)}{t}dt$ $n > 0$ integer

Unilateral Laplace Transform Transform Table [Continued]

	$X(s) = \mathcal{L}_+\{x(t)\}$	$x(t) = \mathcal{L}_+^{-1}\{X(s)\}$
118	$\frac{1}{s}e^{-as}$	$u(t-a)$
119	$\frac{1}{s^2}e^{-as}$	$(t-a)u(t-a)$
120	$\left(\frac{a}{s}+\frac{1}{s^2}\right)e^{-as}$	$tu(t-a)$
121	$\left(\frac{2}{s^3}+\frac{2a}{s^2}+\frac{a^2}{s}\right)e^{-as}$	$t^2u(t-a)$
122	$\frac{1}{s}(e^{-as}-e^{-bs})$ $a<b$	$u(t-a)-u(t-b)$
123	$\left(\frac{1-e^{-s}}{s}\right)^2$	$\begin{cases} t & 0<t\le 1 \\ 2-t & 1<t\le 2 \\ 0 & 2<t \end{cases}$
124	$\left(\frac{1-e^{-s}}{s}\right)^3$	$\begin{cases} \frac{1}{2}t^2 & 0<t\le 1 \\ \frac{3}{4}-(t-\frac{3}{2})^2 & 1<t\le 2 \\ \frac{1}{2}(t-3)^2 & 2<t\le 3 \\ 0 & 3<t \end{cases}$
125	$\frac{1}{s^2}(1-e^{-s})$	$\begin{cases} t & 0<t\le 1 \\ 1 & 1<t \end{cases}$
126	$\frac{1}{s^3}(1-e^{-s})^2$	$\begin{cases} \frac{1}{2}t^2 & 0<t\le 1 \\ 1-\frac{1}{2}(t-2)^2 & 1<t\le 2 \\ 1 & 2<t \end{cases}$
127	$\frac{1}{s(1+e^{-s})}$	$\sum_{k=0}^{\infty}(-1)^k u(t-k)$
128	$\frac{1}{s\sinh s}$	$2\sum_{k=0}^{\infty}u(t-2k-1)$
129	$\frac{1}{s\cosh s}$	$2\sum_{k=0}^{\infty}(-1)^k u(t-2k-1)$
130	$\frac{1}{s}\tanh s$	$u(t)+2\sum_{k=0}^{\infty}(-1)^k u(t-2k)$ or $\sum_{k=0}^{\infty}(-1)^k u(t-2k)u(2k+2-t)$
131	$\frac{e^s-s-1}{s^2(e^s-1)}$	$t-\sum_{k=0}^{\infty}u(t-k)$ or $\sum_{k=0}^{\infty}(t-k)u(t-k)u(k+1-t)$

6.3 The Discrete-Time (Starred) Laplace Transform

While the $\mathcal{Z}$ transform is more compact, a discrete-time version of the Laplace transform known as the **starred Laplace transform** is occasionally used in control systems. The **discrete-time Laplace transform** $\tilde{X}(s)$ of a discrete signal $x[n] = x(nT)$ is defined as

$$\tilde{X}(s) = \mathcal{L}_*\{x[n]\} \sum_{n=-\infty}^{\infty} x[n]e^{-snT}$$

where T corresponds to the discrete-time step size. Note that the discrete-time Laplace transform is periodic in the $j\omega$ direction.

In general, the sum defining the forward discrete-time Laplace transform is convergent only over a band of σ values which define its region of convergence.

The discrete-time Laplace transform is closely related to the discrete-time Fourier transform. In fact, the discrete-time Laplace transform can be viewed as the discrete-time Fourier transform of the signal $x[n]e^{-\sigma nT}$. The discrete-time Laplace transform is related to the $\mathcal{Z}$ transform according to the mapping

$$z = e^{sT} = e^{\sigma T} e^{j\omega T}.$$

This mapping results in the mapping of $j\omega$ axis of the s plane to the unit circle in the z plane. The left-half s plane is mapped to the interior of the unit circle while the right-half s plane is mapped to the exterior of the unit circle.

The **inverse discrete-time Laplace transform** is defined as

$$x[n] = \mathcal{L}_*^{-1}\{\tilde{X}(s)\} = \frac{T}{2\pi j} \int_{\sigma - j\omega_s/2}^{\sigma + j\omega_s/2} X(s)e^{snT}$$

where $\omega_s = 2\pi/T$.

Because the discrete-time Laplace transform is not widely used, transform tables are not provided. The single most important transform relationship is given by

$$e^{\alpha nT}u[nT] \longleftrightarrow \frac{1}{1 - e^{\alpha T}e^{-sT}}, \qquad Re[s] > Re[\alpha].$$

6.4 The $\mathcal{Z}$ Transform

The $\mathcal{Z}$ transform is primarily used on discrete signals (sequences) and systems. It forms the basis of modern digital signal processing.

6.4.1 One-Dimensional Case

In some sense the $\mathcal{Z}$ transform is a generalization of the Discrete Fourier Transform. It provides a techinque to apply complex analysis to discrete systems. The *bilateral* or *two-sided* $\mathcal{Z}$ transform, $X(z) = \mathcal{Z}\{x[n]\}$, of a discrete-time sequence $x[n]$ is defined as

$$X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n}$$

where z a complex variable with $z = re^{j\omega}$ where $r \geq 0$ and ω are real numbers. This infinite power series converges only when

$$\sum_{n=-\infty}^{\infty} |x[n]r^{-n}| < \infty.$$

The **region of convergence** (if it exists) is an annular region $R_{x-} < |z| < R_{x+}$ where $0 \leq R_{x-} < \infty$ and $0 < R_{x+} \leq \infty$. Within the region of convergence (ROC), $X(z)$ is an analytic function which implies that $X(z)$ and all of its derivatives are continuous functions of z. The ROC with be either an open disk, an annulus (donut) or the complement of a closed disk. Note there there may be several (disjoint) ROCs. Also, note that a finite length sequence has a ROC which includes all of the z plan except (possibly) $z = 0$ or $|z| = \infty$.

A tighter requirement on the existence of the $\mathcal{Z}$ transform is that the sequence be absolutely summable, i.e.,

$$\sum_{n=-\infty}^{\infty} |x[n]||z|^{-n} < \infty$$

for z in the ROC.

The **inverse $\mathcal{Z}$ transform** is given by the contour integral,

$$x[n] = \frac{1}{2\pi j} \oint_C X(z)z^{n-1}dz \tag{6.1}$$

where C represents a counterclockwise closed contour contained in the region of convergence of $X(z)$ and including the origin ($z = 0$). This integral is equivalent to the sum of the residues of $X(z)z^{n-1}$ evaluated at the poles of $X(z)$ contained inside of C. Because of possible pole/zero cancelation, the inverse $\mathcal{Z}$ transform is not unique.

To compute the residues, $X(z)z^{n-1}$ (assuming it is a rational function of z), is expressed as

$$X(z)z^{n-1} = \frac{\phi(z)}{(z - z_0)^n}$$

where $X(z)z^{n-1}$ has n poles at $z = z_0$ and $\phi(z)$ has no poles at $z = z_0$. Then the residue is given by,

$$Res[X(z)z^{n-1} \text{ at } z = z_0] = \frac{1}{(n-1)!}\left[\frac{d^{n-1}}{dz^{n-1}}\phi(z)\right]_{z=z_0}$$

In computing the $\mathcal{Z}$ transform of $x[n]$, it is often convenient to decompose $x[n]$ into left-side and right-side sequences, evaluate the regions of convergence and determine the overlap (if any). The overlap in regions of convergence provides the region of convergence of the full $\mathcal{Z}$ transform. A *left-side* sequence is zero for all $n > C > -\infty$ for some constant C while a *right-side* sequence is zero for all $n < C < \infty$.

The region of convergence (ROC) of a left-side sequence is contained in a disk with a radius equal to the smallest magnitue (but non-zero) pole of the $\mathcal{Z}$ transform and may or may not include $z = 0$. The region of convergence of a right-side sequence extends from a circle of radius begining at the largest magnitude (but finite) pole of the $\mathcal{Z}$ transform and extending outward. The ROC may or may not include $|z| = \infty$. A two-sided sequence (which can be expressed as the sum of a left-side and a right-side sequence) has a ROC which is an annular ring (donut) and is the intersection of the ROCs of the left-side and right-side sequences (if such an intersection exists). The ROC is an annular ring bounded by the smallest non-zero and largest finite poles of the $\mathcal{Z}$ transform

6.4.2 Relationship of the Discrete Fourier and the $\mathcal{Z}$ Transforms

By letting $z = e^{j\omega}$ the discrete Fourier transform (DFT) becomes a special case of the $\mathcal{Z}$ transform, i.e., the $\mathcal{Z}$ transform evaluated on the unit circle. However, note that if the region of convergence of the $\mathcal{Z}$ transform *does not* include the unit circle, the DFT does not exist. Examples can be developed for which the $\mathcal{Z}$ transform exists but the DFT does not. Note that because of the common use of DFT as a limit, the DFT of a sequence may "exist" while the $\mathcal{Z}$ transform does not exist! For example, the sequence $x[n] = \cos(an)$ is considered to have a DFT but since $x[n]$ is not absolutely summable, the $\mathcal{Z}$ transform does not exist.

6.4.3 Relationship of the Laplace and $\mathcal{Z}$ Transforms

The $\mathcal{Z}$ transform can be derived from the bilateral (two-sided) Laplace transform by setting $z = e^s$. The $\mathcal{Z}$ transform is also closely related to the discrete-time Laplace transform (see the discussion in the previous section).

6.4.4 Key Properties of the 1-D $\mathcal{Z}$ Transform

Let $X(z)$ and $Y(z)$ be the $\mathcal{Z}$ transforms of $x[n]$ and $y[n]$, respectively, with corresponding regions of convergence of $R_{x^-} < |z| < R_{x^+}$ and $R_{y^-} < |z| < R_{y^+}$. Let a and b be complex constants.

Linearity.

$$\mathcal{Z}\{ax[n] + by[n]\} = aX(z) + bY(z), \quad R_- < |z| < R_+$$

where $R_- = \max\{R^{x^-}, R^{y^-}\}$ and $R_- = \min\{R^{x^-}, R^{y^-}\}$.

Sample Shift.

$$\mathcal{Z}\{x[n+n_0]\} = z^{n_0}X(z), \quad R_{x^-} < |z| < R_{x^+}.$$

Note, however, that time shifting introduces zeros at $z = \infty$ at poles $z = 0$. This may prevent convergence at these points.

Exponential Multiplication.

$$\mathcal{Z}\{a^n x[n]\} = X\left(\frac{z}{a}\right), \quad |a|R_{x^-} < |z| < |a|R_{x^+}.$$

$$\mathcal{Z}\{a^{-n} x[n]\} = X\left(a\right), \quad |a|^{-1}R_{x^-} < |z| < |a|^{-1}R_{x^+}.$$

Differentiation.

$$\mathcal{Z}\{nx[n]\} = -z\frac{d}{dz}X(z), \quad R_{x^-} < |z| < R_{x^+}.$$

$$\mathcal{Z}\{-(n-1)x[n-1]\} = \frac{d}{dz}X(z), \quad R_{x^-} < |z| < R_{x^+}.$$

Note, however, that differentiation introduce zeros at $z = \infty$ at poles $z = 0$. This may prevent convergence at these points.

Conjugation.

$$\mathcal{Z}\{x^*[n]\} = X^*(z^*), \quad R_{x^-} < |z| < R_{x^+}.$$

Convolution.

$$\mathcal{Z}\left\{\sum_{k=-\infty}^{\infty} x(k)y(n-k)\right\} = X(z)Y(z), \quad R_{x^-} < |z| < R_{x^+},\ R_{y^-} < |z| < R_{y^+}.$$

The region of convergence may be larger than the intersection of the individual regions of convergence if there are pole zero cancelations.

$$\begin{aligned}\mathcal{Z}\{x[n]y[n]\} &= \frac{1}{2\pi j}\oint_{C_1} X\left(\frac{z}{p}\right)Y(p)p^{-1}dp, \quad R_{x^-}R_{y^-} < |z| < R_{x^+}R_{y^+} \\ &\quad \text{or} \\ &\quad \frac{1}{2\pi j}\oint_{C_2} X(p)Y\left(\frac{z}{p}\right)p^{-1}dp, \quad R_{x^-}R_{y^-} < |z| < R_{x^+}R_{y^+}\end{aligned}$$

where C_1 is a counterclockwise closed contour within the regions of convergence of $X(z/p)$ and $Y(p)$ and C_2 is a counterclockwise closed contour within the regions of convergence of $X(p)$ and $Y(z/p)$. The region of convergence for $W(z)$ is $R_{x^-}R_{y^-} < |z| < R_{x^+}R_{y^+}$ though it may be larger.

Initial Value Theorem. If $x[n] = 0$ for $n < 0$, then

$$x(0) = \lim_{z\to\infty} X(z).$$

6.4.5 Computing the Inverse $\mathcal{Z}$ Transform

While the $\mathcal{Z}$ transform is a unique function of a given sequence, the inverse $\mathcal{Z}$ transform is not. This is the result of pole/zero cancelations.

Several strategies exists for determing the inverse $\mathcal{Z}$ transform. The primary methods are: table lookup, partial-fraction expansion, and power series expansion. To apply the table lookup method, decompose the $\mathcal{Z}$ transform into known components. The partial-fraction expansion method is based on determing the partial fraction expansion of a rational $\mathcal{Z}$ transform (i.e., writing the $\mathcal{Z}$ transform in simplier terms) to determine the inverse $\mathcal{Z}$ transform. Because the $\mathcal{Z}$ transform is a power series, the inverse can often be determined using a power series expansion. This approach is primarily applicable to a finite length power series.

6.4.6 Two-Dimensional $\mathcal{Z}$ Transform

The two-dimensional $\mathcal{Z}$ transform, $X(z_1, z_2) = \mathcal{Z}\{x[m,n]\}$, of a discrete-time sequence $x[m,n]$ is defined as

$$X(z_1, z_2) = \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} x[m,n] z_1^{-m} z_2^{-n}$$

where z_1 and z_2 are complex variables with $z_1 = r_1 e^{j\omega_1}$ and $z_2 = r_2 e^{j\omega_2}$. This infinite power series converges only when

$$\sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} |x[m,n] r_1^{-m} r_2^{-n}| < \infty.$$

Unlike the one-dimensional case, the **region of convergence** (if it exists) may by a complicated volume. Within the region of convergence, $X(z_1, z_2)$ is an analytic function. Note that when $r_1 = r_2 = 1$ that the 2-D $\mathcal{Z}$ transform is equivalent to the 2-D Fourier transform.

The inverse two-dimensional $\mathcal{Z}$ transform is given by the contour integral,

$$x[m,n] = \frac{1}{(2\pi j)^2} \oint_{C_1} \oint_{C_1} X(z_1, z_2) z_1^{m-1} z_2^{n-1} dz_1 dz_2 \tag{6.2}$$

where C_1 are closed contours contained within the region of convergence of $X(z_1, z_2)$ and which include the origin ($z_1 = z_2 = 0$). Computing the multidimensional inverse $\mathcal{Z}$ transform can be difficult since there is no equivalent multidimensional version of the residue theorem.

6.4.7 Key Properties of the 2-D $\mathcal{Z}$ Transform

Let $X(z_1, z_2)$ and $Y(z_1, z_2)$ be the $\mathcal{Z}$ transforms of $x[m,n]$ and $y[m,n]$, respectively. Regions of convergence are problem dependent.

Linearity.

$$\mathcal{Z}\{ax[m,n] + by[m,n]\} = aX(z_1, z_2) + bY(z_1, z_2).$$

Sample Shift.

$$\mathcal{Z}\{x[m+m_0,n+n_0]\} = z_1^{m_0} z_2^{n_0} X(z_1,z_2).$$

Exponential Multiplication.

$$\mathcal{Z}\{a^m b^n x[m,n]\} = X(\frac{z_1}{a},\frac{z_2}{b}).$$

Differentiation.

$$\mathcal{Z}\{mnx[m,n]\} = \frac{d^2}{dz_1 dz_2} X(z_1 z_2).$$

Note that time shifting introduces zeros at infinity at poles at the origin.
Conjugation.

$$\mathcal{Z}\{x^*[m,n]\} = X^*(z_1^*,z_2^*).$$

Convolution.

$$\mathcal{Z}\{\sum_{k=-\infty}^{\infty}\sum_{l=-\infty}^{\infty} x[k,l]y[m-k,n-l]\} = X(z_1,z_2)Y(z_1,z_2).$$

$$\mathcal{Z}\{x[m,n]y[m,n]\} = \frac{1}{4\pi^2}\oint_{C_1}\oint_{C_2} X\left(\frac{z_1}{p_1},\frac{z_2}{p_2}\right) Y(p_1,p_2) p_1^{-1} p_2^{-1} dp_1 dp_2$$

where C_1 and C_2 are closed contour within the regions of convergence enclosing the origin.

6.5 $\mathcal{Z}$ Transform Tables

6.5.1 One-Dimensional $\mathcal{Z}$ Transform

Definition and Key Properties

	$x[n] = \mathcal{Z}^{-1}\{X(z)\}$	$X(z) = \mathcal{Z}\{x[n]\}$	Region of Convergence
1	$x[n]$	$\sum_{n=-\infty}^{\infty} x[n]z^n$	$R_{x-} < \lvert z\rvert < R_{x+}$
2	$ax[n] + by[n]$	$aX(z) + bY(z)$	$\max\{R^{x^-}, R^{y^-}\} < \lvert z\rvert < \min\{R^{x^-}, R^{y^-}\}$
3	$x[n+n_0]$	$z^{n_0}X(z)$	$R_{x-} < \lvert z\rvert < R_{x+}$
4	$a^n x[n]$	$X\left(\frac{z}{a}\right)$	$\lvert a\rvert R_{x-} < \lvert z\rvert < \lvert a\rvert R_{x+}$
5	$nx[n]$	$-z\frac{d}{dz}X(z)$	$R_{x-} < \lvert z\rvert < R_{x+}$
6	$x^*[n]$	$X^*(z^*)$	$R_{x-} < \lvert z\rvert < R_{x+}$
7	$x[-n]$	$X(\frac{1}{z})$	$1/R_{x-} < \lvert z\rvert < 1/R_{x+}$
8	$x[n)] * y[n]$	$X(z)Y(z)$	$\max\{R^{x^-}, R^{y^-}\} < \lvert z\rvert < \min\{R^{x^-}, R^{y^-}\}$
9	$x[n]y[n]$	$\frac{1}{2\pi j}\oint_{C_1} X\left(\frac{z}{p}\right)Y(p)p^{-1}dp$	$R_{x-}R_{y-} < \lvert z\rvert < R_{x+}R_{y+}$
		$\frac{1}{2\pi j}\oint_{C_2} X(p)Y\left(\frac{z}{p}\right)p^{-1}dp$	$R_{x-}R_{y-} < \lvert z\rvert < R_{x+}R_{y+}$
10	$Re[x[n]]$	$\frac{1}{2}[x(z) + X^*(z^*)]$	$R_{x-} < \lvert z\rvert < R_{x+}$
11	$Im[x[n]]$	$\frac{1}{2j}[x(z) - X^*(z^*)]$	$R_{x-} < \lvert z\rvert < R_{x+}$
12	$x[n+k]u[n]$	$z^k X(z) - z^k \sum_{i=0}^{k-1} x(i)z^{-i}$	depends on $x[n]$
13	$x[n-k]u[n]$	$z^{-k}X(z) + \sum_{i=1}^{k} x(-i)z^{-k+i}$	depends on $x[n]$
14	$x[n-k]u[n-k]$	$z^{-k}X(z)$	depends on $x[n]$
15	$n^a x[n]$	$-z\frac{d^a}{dz^a}X(z)$	$R_{x-} < \lvert z\rvert < R_{x+}$

1-D $\mathcal{Z}$ Transform Limits

$$\lim_{n \leftarrow 0} f[n] = \lim_{z \leftarrow \infty} F(z)$$

$$\lim_{n \leftarrow \infty} f[n] = \lim_{z \leftarrow 1}(1 - z^{-1})F(z)$$

1-D $\mathcal{Z}$ Transform Transform Table

1-D $\mathcal{Z}$ Transform Transform Table

	$x[n] = \mathcal{Z}^{-1}\{X(z)\}$	$X(z) = \mathcal{Z}\{x[n]\}$	Region of Convergence
1	$\delta[n]$	1	$0 \le \lvert z\rvert \le \infty$
2	$\delta[n-m]$	z^{-m}	$m > 0: \quad 0 < \lvert z\rvert \le \infty$
			$m < 0: \quad 0 \le \lvert z\rvert < \infty$
3	$u[n]$	$1/(1-z^{-1}) = z/(z-1)$	$\lvert z\rvert > 1$
4	$u[-n-1]$	$-1/(1-z^{-1}) = -z/(z-1)$	$\lvert z\rvert < 1$
5	$a^n u[n]$	$1/(1-az^{-1}) = z/(z-a)$	$\lvert z\rvert > \lvert a\rvert$
6	$na^n u[n]$	$az^{-1}/(1-az^{-1})^2$	$\lvert z\rvert > \lvert a\rvert$
7	$a^n u[-n-1]$	$-z/(z-a)$	$\lvert z\rvert < a$
8	$na^n u[-n-1]$	$-az^{-1}/(1-az^{-1})^2$	$\lvert z\rvert < \lvert a\rvert$
9	$\begin{cases} a^n & 0 \le n \le N-1 \\ 0 & \text{else} \end{cases}$	$(1-a^N z^N)/(1-az^{-1})$	$0 < \lvert z\rvert < \infty$
10	$\begin{pmatrix} n \\ m \end{pmatrix} a^{n-m} u(-n-1)$	$+z/(z-a)^{m+1}$	$\lvert z\rvert > a$
		$-z/(z-a)^{m+1}$	$\lvert z\rvert < a$
11	e^{an}	$\dfrac{1}{1-e^a z^{-1}}$	
12	an	$\dfrac{az^{-1}}{(1-z^{-1})^2}$	
13	$(an)^2$	$\dfrac{a^2 z^{-1}(1-z^{-1}}{(1-z^{-1})^3}$	
14	$(an)^3$	$a^3 \dfrac{3z^{-2}(1-z^{-1}}{(1-z^{-1})^4} +$	
		$a^3 \dfrac{z^{-1}(1-2z^{-1}}{(1-z^{-1})^3}$	
15	$[\sin an]u[n]$	$\dfrac{1-z^{-1}\cos a}{1-2z^{-1}\cos a + z^{-2}}$	$\lvert z\rvert > 1$
16	$[\cos an]u[n]$	$\dfrac{z^{-1}\sin a}{1-2z^{-1}\cos a + z^{-2}}$	$\lvert z\rvert > 1$
17	$[r^n \sin an]u[n]$	$\dfrac{1-z^{-1} r\cos a}{1-2z^{-1} r\cos a + r^2 z^{-2}}$	$\lvert z\rvert > r$
18	$[r^n \cos an]u[n]$	$\dfrac{z^{-1} r\sin a}{1-2z^{-1} r\cos a + r^2 z^{-2}}$	$\lvert z\rvert > r$
19	$na^{n-1}u[n]$	$z/(z-a)^2$	$\lvert z\rvert > a$

1-D $\mathcal{Z}$ Transform Transform Table [continued]

	$x[n] = \mathcal{Z}^{-1}\{X(z)\}$	$X(z) = \mathcal{Z}\{x[n]\}$	Region of Convergence
20	$\cosh an$	$\dfrac{1 - z^{-1}\cosh a}{1 - z^{-1}2\cosh a + z^{-2}}$	
21	$\sinh an$	$\dfrac{z^{-1}\sinh a}{1 - z^{-1}2\cosh a + z^{-2}}$	
22	$u(n) - u(n-2)$	$1 + z^{-1}$	
23	$\dfrac{1}{a-b}[a^n - b^n]$	$\dfrac{z}{(z-a)(z-b)}$	
24	$\dfrac{1}{n!}$	$e^{1/z}$	
25	$\dfrac{1}{(2n)!}$	$\cosh z^{1/2}$	

6.5.2 Two-Dimensional $\mathcal{Z}$ Transform

6.5.3 Definition and Key Properties

	$x[m,n] = \mathcal{Z}^{-1}\{X(z_1,z_2)\}$	$X(z_1,z_2) = \mathcal{Z}\{x[m,n]\}$
1	$x(m,n)$	$\sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty} x[m,n] z_2^m z_2^n$
2	$ax[m,n] + by[m,n]$	$aX(z_1,z_2) + bY(z_1,z_2)$
3	$x[m+m_0, n+n_0]$	$z_1^{m_0} z_2^{n_0} X(z_1,z_2)$
4	$a^m b^n x[m,n]$	$X\left(\frac{z_1}{a}, \frac{z_2}{b}\right)$
5	$mnx[m,n]$	$\frac{d^2}{dz_1 dz_2} X(z_1 z_2)$
6	$x^*[m,n]$	$X^*(z_1^*, z_2^*)$
7	$x[-m,-n]$	$X(z_1^{-1}, z_2^{-1})$
8	$x[m,-n]$	$X(z_1, z_2^{-1})$
9	$x[-m,n]$	$X(z_1^{-1}, z_2)$
10	$x[m,n] * y[m,n]$	$X(z_1,z_2)Y(z_1,z_2)$
11	$x[m,n]y[m,n]$	$\frac{1}{4\pi^2}\oint_{C_1}\oint_{C_2} X\left(\frac{z_1}{p_1}, \frac{z_2}{p_2}\right) Y(p_1,p_2)\, p_1^{-1} p_2^{-2} dp_1 dp_2$
12	$Re[x[m,n]]$	$\frac{1}{2}[x(z_1,z_2) + X^*(z_1^*, z_2^*)]$
13	$Im[x[m,n]]$	$\frac{1}{2j}[x(z_1,z_2) - X^*(z_1^*, z_2^*)]$
14	$x[m]y[n]$	$X(z_1)Y(z_2)$

2-D $\mathcal{Z}$ Transform Transform Table

2-D $\mathcal{Z}$ Transform Transform Table

	$x[m,n] = \mathcal{Z}^{-1}\{X(z_1,z_2)\}$	$X(z_1,z_2) = \mathcal{Z}\{x[m,n]\}$
1	$\delta[m,n]$	1
2	$\delta[m-m_0, n-n_0]$	$z_1^{-m_0} z_2^{-n_0}$
3	$u[m]u[n]$	$\frac{1}{(1-z_1^{-1})(1-z_2^{-1})}$
4	$a^m b^n u[m]u[n]$	$\frac{z_1 z_2}{(z_1-a)(z_2-b)}$

6.6 Partial Fraction Expansion

Partial fraction expansion is a technique to convert a rational polynomial in standard form to a sum of low-order polynomials. Let $H(s)$ be a rational polynomial s of the form

$$H(s) = \frac{\displaystyle\sum_{k=0}^{m} b_m s^m}{s^n + \displaystyle\sum_{l=0}^{n-1} a_l s^l}.$$

If $m \geq n$ synthetic division is performed to transform $H(s)$ into the form

$$H(s) = \sum_{i=0}^{m-n} c_i s^i + H'(s)$$

where $H'(s)$ is also a rational polynomial with the order of the numerator (m) is less than the order of the denominator (n). Partial fraction expansion is then continued using $H'(s)$ in place of the following.

Given $H(s)$ with $m < n$, the denominator is factored into simple roots, i.e.,

$$s^n + \sum_{l=0}^{n-1} a_l s^l = \prod_{l=0}^{n} (s - \rho_l).$$

If all the roots are distinct, $H(s)$ can be expanded as

$$H(s) = \sum_{i=0}^{n} \frac{A_i}{(s - \rho_i)}$$

where

$$A_i = \left[(s - \rho_i) H(s)\right]\Big|_{s=\rho_i}.$$

If there are multiple roots, i.e., $\rho_i = \rho_j$ for some i and j, the denominator can be expressed as

$$s^n + \sum_{l=0}^{n-1} a_l s^l = \prod_{j=0}^{M} (s - \rho_j)^{N_j}$$

where M is the number of distinct roots and N_j is the order of each root. $H(s)$ can then be expanded as,

$$H(s) = \sum_{i=0}^{M} \sum_{i=0}^{N_j} \frac{A_{ij}}{(s - \rho_i)^j}$$

where

$$A_{ij} = \frac{1}{(N_j - j)!} \left\{ \frac{d^{(N_j - j)}}{ds^{(N_j - j)}} \left[(s - \rho_i)^j H(s)\right] \right\}\Bigg|_{s=\rho_i}.$$

Chapter 7

Related Transforms

7.1 Cosine and Sine Transforms

There are a number of different definitions of the Cosine and Sine transforms. A common definition for the cosine transform $\mathcal{C}\{f(t)\} = F_C(\omega)$ is

$$F_C(\omega) = \int_0^\infty f(t)\cos\omega t dt.$$

The inverse cosine transform is

$$f(t) = \frac{2}{\pi}\int_0^\infty F_C(\omega)\cos\omega t d\omega.$$

The corresponding definition for the sine transform $\mathcal{S}\{f(t)\} = F_S(\omega)$ is

$$F_S(\omega) = \int_0^\infty f(t)\sin\omega t dt.$$

The inverse cosine transform is

$$f(t) = \frac{2}{\pi}\int_0^\infty F_S(\omega)\sin\omega t d\omega.$$

The Cosine and Sine transforms are closely related to the Fourier Transform. Note that the Fourier transform $F(\omega)$ of a signal $f(t)$ where $f(t) = 0$ for $t < 0$ can be expressed in terms of the Cosine and Sine transforms as

$$F(\omega) = F_C(\omega) + jF_S(\omega).$$

7.1.1 Key Properties of the Cosine and Sine Transforms

7.1.2 Cosine Transform Table

In the following table the constant a is a positive number.

	$f(t)$ for $t \geq 0$ ($f(t) = 0 for t < 0$)	$F_C(\omega)$ for $\omega \geq 0$
1	$f(t) = \begin{cases} 1 & t \leq a \\ 0 & t > a \end{cases}$	$\dfrac{\sin a\omega}{\omega}$
2	e^{-at}	$\dfrac{a}{a^2 + \omega^2}$
3	e^{-at^2}	$\dfrac{1}{2}\sqrt{\pi/a}e^{-\omega^2/4a}$

7.1.3 Sine Transform Table

In the following table the constant a is a positive number.

	$f(t)$ for $t \geq 0$ ($f(t) = 0 for t < 0$)	$F_S(\omega)$ for $\omega \geq 0$
1	$f(t) = \begin{cases} 1 & t \leq a \\ 0 & t > a \end{cases}$	$\dfrac{1 - \cos a\omega}{\omega}$
2	e^{-at}	$\dfrac{\omega}{a^2 + \omega^2}$
3	te^{-at^2}	$\sqrt{\pi/a}\dfrac{\omega}{4a}e^{-\omega^2/4a}$

7.2 Hankel Transform

The p^{th} order Hankel transform $F_p(s) = \mathcal{H}_p\{f(t)\}$ of the one sided ($t \geq 0$) signal $f(t)$ is defined as,

$$F_p(s) = \mathcal{H}_p\{f(t)\} = \int_0^\infty tf(t)J_p(st)\,dt$$

where $J_p(x)$ is a Bessal function of the first kind of order p,

$$J_p(x) = \left(\frac{x}{2}\right)^p \sum_{k=0}^{\infty} \frac{\left(\frac{1}{4}x^2\right)^k}{k!\,\Gamma(p+k+1)}$$

where the gamma function (for integer arguments) is

$$\Gamma(m) = m!$$

For non-integers and for the real part of z greater than zero,

$$\Gamma(z) = \int_0^\infty x^z e^{-x} dx$$

and $\Gamma(z+1) = z\Gamma(z+1)$.

For $p > -1/2$, the inverse Hankel transform is

$$f(t) = \mathcal{H}_p^{-1}\{F_p(s)\} = \int_0^\infty sF_p(s)J_p(st)\,ds.$$

7.3 Hartley Transform

The Hartley transform is a reciporocal (the forward and inverse transforms are identical) transform related to the Fourier transform. The Hartley transform $\mathcal{H}_a\{f(t)\} = F_H(\omega)$ of $f(t)$ is defined as,

$$\mathcal{H}_a\{f(t)\} = F_H(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t)(\cos\omega t + \sin\omega t)dt$$

where the integral exists. In this text $f(t)$ is assumed to be real though this may be generalized. Note that

$$\mathcal{H}_a^{-1}\{F_H(\omega)\} = f(t).$$

Given the Hartely transform $F_H(\omega)$ of $f(t)$ the Fourier transform $F(\omega)$ can be easily generated using reflections and additions, i.e.,

$$F(\omega) = \text{Even}[F_H(\omega)] - j\,\text{Odd}[F_H(\omega)]$$

and

$$F_H(\omega) = Re[F(\omega)] - Im[F(\omega)].$$

Using these relationships, the key properties of the Hartley transform can be derived from the key properties of the Fourier transform.

7.4 Hilbert Transform

The Hilbert transform $\mathcal{H}\{f(t)\}$ of $f(t)$ is defined as,

$$\mathcal{H}\{f(t)\} = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{f(\tau)}{\tau - t} d\tau$$

where at $\tau = t$, the Cauchy pinciple value of the integral is used, i.e.,

$$\mathcal{H}\{f(t)\} = \frac{1}{\pi} \lim_{\epsilon \longrightarrow 0} \left[\int_{-\infty}^{t-\epsilon} \frac{f(\tau)}{\tau - t} d\tau + \int_{t+\epsilon}^{\infty} \frac{f(\tau)}{\tau - t} d\tau \right],$$

to avoid the problem at $t = \tau$.

The Hilbert transform is its own inverse, i.e.,

$$\mathcal{H}\{\mathcal{H}\{f(t)\}\} = f(t).$$

The Hilbert transform of $f(t)$ may be obtained by a linear filtering operation with the function $h(t) = -1/(\pi t)$, i.e.,

$$\mathcal{H}\{f(t)\} = \frac{-1}{\pi t} * f(t).$$

Note that since $-1/(\pi t) \longleftrightarrow j\text{sgn}\,\omega$, the Hilbert transform may be defined in terms of the Fourier transform as,

$$\mathcal{H}\{f(t)\} = \mathcal{F}^{-1}\{j\text{sgn}\,\omega\,\mathcal{F}\{f(t)\}\}.$$

The Hilbert transform is unique to within a constant since the Hilbert transform of a constant is zero.

7.4.1 Key Properties of the Hilbert Transform

A key property of a real function $f(t)$ which is zero for $t < 0$ is that the real and imaginary parts of its Fourier transform form a Hilbert transform pair, i.e.,

$$\mathcal{H}\{Re[\mathcal{F}\{f(t)\}]\} \longleftrightarrow Im[\mathcal{F}\{f(t)\}]$$

or,

$$\mathcal{F}\{f(t)\} = Re[\mathcal{F}\{f(t)\}] + j\mathcal{H}\{Re[\mathcal{F}\{f(t)\}]\}.$$

The Hilbert transform has the following properties:

$f(t)$	$\mathcal{H}\{f(t)\}$
real, even	real, odd
real, odd	real, even
imag, even	imag, odd
imag, odd	imag, even

Let $f(t) \overset{\mathcal{H}}{\longleftrightarrow} \mathcal{H}\{f(t)\} = F(t)$ and $g(t) \overset{\mathcal{H}}{\longleftrightarrow} \mathcal{H}\{g(t)\} = G(t)$ denote Hilbert transform pairs. It can easily be shown that the Hilbert transform is **linear**, i.e.,

$$\mathcal{H}\{af(t) + bg(t)\} = aF(t) + bG(t)$$

where a and b are any constants.

Similarity is exhibited by the Hilbert transform, i.e.,

$$g(at) \overset{\mathcal{H}}{\longleftrightarrow} g(at).$$

The **Shifting** theorem for Hilbert transforms state that

$$g(t-\tau) \longleftrightarrow G(t-\tau).$$

Power is preserved by the Hilbert transform.

$$\int_{-\infty}^{\infty} g(t)g^*(t)dt \overset{\mathcal{H}}{\longleftrightarrow} \int_{-\infty}^{\infty} G(t)G^*(t)dt$$

as is the **autocorrelation**

$$\int_{-\infty}^{\infty} g^*(t)g(t-\tau)dt \overset{\mathcal{H}}{\longleftrightarrow} \int_{-\infty}^{\infty} G^*(t)G(t-\tau)dt.$$

For **convolution**, the following properties apply:

$$f(t) * g(t) \overset{\mathcal{H}}{\longleftrightarrow} -F(t) * G(t)$$

$$\mathcal{H}\{f(t) * g(t)\} = F(t) * g(t) = f(t) * G(t).$$

7.4.2 Hilbert Transform Tables

Definition and Key Properties

	$g(t) = \mathcal{H}\{G(t)\}$	$\mathcal{H}\{g(t)\} = G(t)$
1	$g(t)$	$\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{f(\tau)}{\tau - t}d\tau$
2	$\mathcal{H}\{g(t)\}$	$g(t)$
3	$g(t)$	$\mathcal{F}^{-1}\{j\,\mathrm{sgn}\,\omega\,\mathcal{F}\{g(t)\}\}$
4	$af(t) + bg(t)$	$aF(t) + bG(t)$
5	$g(at)$	$G(at)$
6	$g(t-\tau)$	$G(t-\tau)$
7	$\int_{-\infty}^{\infty} g(t)g^*(t)dt$	$\int_{-\infty}^{\infty} G(t)G^*(t)dt$
8	$\int_{-\infty}^{\infty} g^*(t)g(t-\tau)dt$	$\int_{-\infty}^{\infty} G^*(t)G(t-\tau)dt$
9	$f(t) * g(t)$	$-F(t) * G(t)$
10	$f(t) * g(t)$	$F(t) * g(t)$ or $f(t) * G(t)$
11	constant	0

7.4.3 Hilbert Transform Table

Hilbert Transform Table

	$g(t) = \mathcal{H}\{G(t)\}$	$\mathcal{H}\{g(t)\} = G(t)$
1	$g(t)$	$\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{f(\tau)}{\tau - t}d\tau$
2	$\cos at$	$\sin at$
3	$\sin at$	$-\cos at$
4	$\frac{\sin at}{t}$	$\frac{\cos at - 1}{t}$
5	$\Pi(t)$	$\frac{1}{\pi}\ln 6\left\|\frac{t-1/2}{t+1/2}\right\|$
7	$\delta(t)$	$-1/(\pi t)$
8	$\frac{1}{1+t^2}$	$-\frac{t}{1+t^2}$
9	$\frac{d}{dt}\mathrm{sinc}\,t$	$-\pi\mathrm{sinc}\,t - \frac{1}{2}\pi\mathrm{sinc}^2\,\frac{1}{2}t$
10	$\frac{1}{1+t^2}$	$\frac{t}{1+t^2}$
11	$\frac{1-\cos at}{\pi t}$	$-\frac{\sin at}{\pi t}$

Chapter 8

Applications

8.0.1 Windowing

Table 8.1 lists a number of common used discrete time data windows and their properties. In this table, the window function $w(n)$ is defined for $0 \leq n \leq N/$ where N is the window length and is zero elsewhere. An extensive treatment of various windows is given by Harris [9].

Continuous windows are given in Table 8.2. The window length is L so that the window is defined for $0 \leq t \leq L$.

Table 8.1: Some Common Discrete Signal Processing Windows

Window	Formula	Mainlobe Width	First Sidelobe	Rolloff Rate
Rectangular	$w(n) = 1$			
Hamming	$w(n) = \frac{25}{46} + \left(1 - \frac{25}{46}\right)\cos(2\pi n/N)$			
Generalized Hamming	$w(n) = \alpha + (1-\alpha)\cos(2\pi n/N)$ $0 \leq \alpha < 1$			
Hann	$w(n) = 0.5 + 0.5\cos(2\pi n/N)$			
$\cos^m$	$\cos^m(2\pi n/N)$			

Table 8.2: Some Common Continuous Signal Processing Windows

Window	Formula	Mainlobe Width	First Sidelobe	Rolloff Rate
Rectangular	$w(t) = 1$			
Hamming	$w(t) = \frac{25}{46} + \left(1 - \frac{25}{46}\right)\cos(2\pi t)$			
Generalized Hamming	$w(t) = \alpha + (1-\alpha)\cos(2\pi t/L)$ $0 \leq \alpha < 1$			
Hann	$w(t) = 0.5 + 0.5\cos(2\pi t/L)$			
$\cos^m$	$\cos^m(2\pi t/L)$			

Chapter 9

Useful Identities and Facts

9.1 Trigonometric Functions

$\tan\alpha = \sin\alpha/\cos\alpha$
$\cot\alpha = \cos\alpha/\sin\alpha$
$\csc\alpha = 1/\sin\alpha$
$\sec\alpha = 1/\cos\alpha$
$\cos^2\alpha + \sin^2\alpha = 1$

$\sin(\alpha \pm \beta) = \sin\alpha\cos\beta \pm \cos\alpha\sin\beta$
$\cos(\alpha \pm \beta) = \cos\alpha\cos\beta \mp \sin\alpha\sin\beta$

$\sin\alpha\sin\beta = \frac{1}{2}[\cos(\alpha-\beta) - \cos(\alpha+\beta)]$
$\cos\alpha\cos\beta = \frac{1}{2}[\cos(\alpha-\beta) + \cos(\alpha+\beta)]$
$\sin\alpha\cos\beta = \frac{1}{2}[\sin(\alpha+\beta) + \sin(\alpha-\beta)]$
$\cos\alpha\sin\beta = \frac{1}{2}[\sin(\alpha+\beta) - \sin(\alpha-\beta)]$

$\sin(\alpha+\beta)\sin(\alpha-\beta) = \sin^2\alpha - \sin^2\beta = \cos^2\beta - \cos^2\alpha$
$\cos(\alpha+\beta)\cos(\alpha-\beta) = \cos^2\alpha - \sin^2\beta = \cos^2\beta - \sin^2\alpha$
$\sin\alpha \pm \sin\beta = 2\sin\frac{1}{2}(\alpha\pm\beta)\cos\frac{1}{2}(\alpha\mp\beta)$
$\cos\alpha + \cos\beta = 2\cos\frac{1}{2}(\alpha+\beta)\cos\frac{1}{2}(\alpha-\beta)$
$\cos\alpha - \cos\beta = -2\sin\frac{1}{2}(\alpha+\beta)\sin\frac{1}{2}(\alpha-\beta)$

$\sin 2\alpha = 2\sin\alpha\cos\alpha$
$\cos 2\alpha = 2\cos^2\alpha - 1 = 1 - 2\sin^2\alpha = \cos^2\alpha - \sin^2\alpha$
$\sin 3\alpha = 3\sin\alpha - 4\sin^3\alpha$
$\cos 3\alpha = 4\cos^3\alpha - 3\cos\alpha$
$\sin n\alpha = 2\sin(n-1)\alpha\cos\alpha - \sin(n-2)\alpha$
$\cos n\alpha = 2\cos(n-1)\alpha\cos\alpha - \cos(n-2)\alpha$

$\sin^2\alpha = \frac{1}{2}(1-\cos 2\alpha)$
$\cos^2\alpha = \frac{1}{2}(1+\cos 2\alpha)$

$\sin^3 \alpha = \frac{1}{4}(3\sin\alpha - \sin 3\alpha)$
$\cos^3 \alpha = \frac{1}{4}(3\cos\alpha + \cos 3\alpha)$

$$\tan(\alpha + \beta) = \frac{\tan\alpha \pm \tan\beta}{1 \pm \tan\alpha\tan\beta}$$

9.2 Hyperbolic Functions

$\sinh x = \frac{1}{2}(e^x - e^{-x})$
$\cosh x = \frac{1}{2}(e^x + e^{-x})$
$\tanh x = \frac{\sinh x}{\cosh x} = \frac{1-e^{-2x}}{1+e^{-2x}} = \frac{e^x - e^{-x}}{e^x + e^{-x}}$
$\sinh^2 x + \cosh^2 x = 1$
$j\sinh x = \sin jx \quad j\cosh x = \cos jx$
$\text{arcsinh} x = \ln(x + \sqrt{1+x^2}), \quad -\infty < x < \infty$
$\text{arctanh} x = \frac{1}{2}\ln\frac{1+x}{1-x}, \quad -1 < x < 1$

$\sinh(\alpha \pm \beta) = \sinh\alpha\cosh\beta \pm \cosh\alpha\sinh\beta$
$\cosh(\alpha \pm \beta) = \cosh\alpha\cosh\beta \pm \sinh\alpha\sinh\beta$
$\tanh(\alpha \pm \beta) = \frac{\tanh\alpha \pm \tanh\beta}{1 \pm \tanh\alpha\tanh\beta}$

$\sinh 2\alpha = 2\sinh\alpha\cosh\alpha$
$\cosh 2\alpha = 1 + 2\sinh^2\alpha = 2\cosh^2\alpha - 1$
$\cosh^2\alpha - \sinh^2\alpha = 1$
$\tanh^2\alpha + \sec^2\alpha = 1$

$\sinh\alpha \pm \sinh\beta = 2\sinh\frac{1}{2}(\alpha \pm \beta)\cosh\frac{1}{2}(\alpha \mp \beta)$
$\cosh\alpha + \cosh\beta = 2\cosh\frac{1}{2}(\alpha + \beta)\cosh\frac{1}{2}(\alpha - \beta)$
$\cosh\alpha - \cosh\beta = -2\sinh\frac{1}{2}(\alpha + \beta)\sinh\frac{1}{2}(\alpha - \beta)$

$2\cosh\alpha\cosh\beta = \cosh(\alpha + \beta) + \cosh(\alpha + \beta)$
$2\sinh\alpha\sinh\beta = \cosh(\alpha + \beta) - \cosh(\alpha + \beta)$
$2\sinh\alpha\sinh\beta = \cosh(\alpha + \beta) - \cosh(\alpha + \beta)$

9.3 Series

$$\sum_{k=0}^{N} z^k = \frac{z^{N+1} - 1}{z - 1}$$
$$\sum_{k=1}^{N} z^k = \frac{z^{N+1} - z}{z - 1}$$

$$\sum_{k=0}^{\infty} z^k = \frac{1}{1-z}, \qquad |z| < 1$$
$$\sum_{k=0}^{\infty} k z^k = \frac{z}{(1-z)^2}, \qquad |z| < 1$$
$$\sum_{k=1}^{N} k = \frac{N(N+1)}{2}$$
$$\sum_{k=1}^{N} k^2 = \frac{N(N+1)(2N+1)}{6}$$
$$\sum_{k=0}^{\infty} \frac{1}{(k+1)}(-1)^k = \pi/4$$
$$\sum_{k=1}^{\infty} k^{-2} = \pi^2/6$$
$$\sum_{k=0}^{\infty} (2k+1)^{-2} = \pi^2/8$$
$$\sum_{k=0}^{\infty} (2k+1)^{-2} = \pi^2/8$$
$$\sum_{k=1}^{\infty} (-k)^{-2} = \pi^2/12$$

9.3.1 Binomial Series

$$\sum_{k=0}^{n} \binom{n}{k} z_1^k z_2^{n-k} = (z_1 + z_2)^n$$
$$\sum_{k=0}^{n} \binom{n+k}{k} z^k = \frac{1}{(1-z)^{n+1}}$$

9.3.2 Taylor Series

$$f(x+\Delta x) = f(x) + f'(x)\Delta x/1! + f''(x)\frac{(\Delta x)^2}{2!} + \cdots + \frac{f^{(n-1)}(x)(\Delta x)^{n-1}}{(n-1)!} + R_n(\Delta x)$$

where

$$R_n(\Delta x) = f^{(n)}(x + \theta \Delta x)\frac{(\Delta x)^n}{n!}, \qquad 0 < \theta < 1.$$

9.3.3 Maclaurin Series

$$f(x) = f(0) + x f'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + \cdots + \frac{x^{n-1}}{(n-1)!} f^{(n-1)}(0) + R_n$$

where

$$R_n = \frac{x^n}{n!} f^{(n)}(\theta x), \qquad 0 < \theta < 1.$$

9.3.4 Exponential and Logarithmic Series

$$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}$$

$$a^x = \sum_{k=0}^{\infty} \frac{x^k k \ln a}{k!}, \qquad a > 0$$

$$\ln x = \sum_{k=1}^{\infty} \frac{1}{k} \left(\frac{x-1}{x} \right), \qquad x > 1/2$$

$$\ln \frac{1}{1-x} = \sum_{k=1}^{\infty} \frac{x^k}{k}, \qquad -1 \le x \le 1$$

$$\ln(1+x) = \sum_{k=1}^{\infty} (-1)^{k-1} \frac{x^k}{k}, \qquad -1 \le x \le 1$$

$$\ln \frac{1}{2} = \sum_{k=1}^{\infty} \frac{(-1)^k}{k}$$

$$e^x = \sum_{k=0}^{\infty} \frac{z^k}{k!}$$

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!} x^2 + \frac{n(n-1)(n-2)}{3!} x^3 + \cdots$$

$$\text{sinh} x = \sum_{k=0}^{\infty} \frac{x^{2k+1}}{(2k+1)!}$$

$$\text{cosh} x = \sum_{k=0}^{\infty} \frac{x^{2k}}{(2k)!}$$

9.3.5 Trigonometric Series

$$\cos x = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k}}{(2k)!}$$

$$\sin x = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{(2k+1)!}$$

$$\text{tan}\, x = \sum_{k=0}^{\infty} \frac{x^{2k+1}}{(2k+1)!}$$

$$\sin^{-1} x = x + \frac{1}{2 \cdot 3} x^3 + \frac{1 \cdot 3}{2 \cdot 4 \cdot 5} x^5 + \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6 \cdot 7} x^7 + \cdots$$

$$\cos^{-1} x = \pi/2 - \sin^{-1} x$$

$$\text{tan}^{-1} = \begin{cases} \displaystyle\sum_{i=0}^{\infty} \frac{(-1)^i x^{2i+1}}{2i+1} & |x| < 1 \\ \displaystyle\frac{\pi}{2} - \sum_{i=0}^{\infty} \frac{(-1)^i}{(2i+1) x^{2i+1}} & x > 1 \end{cases}$$

$$\sqrt{x} = 1 - \frac{(x-1)}{2} + \frac{(x-1)^2}{2 \cdot 4} - \cdots$$

9.3.6 Riemann's Zeta Function

$$\zeta(p) = \sum_{n=1}^{\infty} \frac{1}{n^p}$$

Note that $\zeta(2) = \pi^2/2$.

9.4 Probability Densities

The characteristic function $\Psi_x(\omega)$ of a probability density $p_x(x)$ is defined as,

$$\Psi_x(\omega) = E[e^{j\omega x}] = \int_{-\infty}^{\infty} e^{j\omega x} p_x(x) dx$$

where $E[\cdot]$ denotes expectation. Note that the characteristic function is the inverse Fourier transform of the probability density $p_x(x)$. The mth moment $E[x^m]$, if it exists, is given by,

$$E[x^m] = (-j)^m \frac{d^m}{d\omega^m} \Psi_x(\omega) \bigg|_{\omega=0} \qquad m > 0$$

Table 9.1 lists the properties and characteristic functions of a number of common probability functions. Continuous densities are denoted by lower case $p(x)$ while discrete distributions use capital $P(n)$.

Table 9.1: Common probability densities and characteristic functions

Name	Probability Distribution	Mean	Variance	Characteristic Function
Gaussian	$p(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-(x-\mu)^2/2\sigma^2}$ $-\infty < x < \infty \quad \sigma > 0 \quad \infty < \mu < \infty$	μ	σ^2	$e^{j\omega\mu - \sigma^2\omega^2/2}$
Uniform	$p(x) = \frac{1}{b-a} \quad a < x < b$ $-\infty < a < b < \infty$	$(a+b)/2$	$(b-a)^2/12$	$j\frac{e^{j\omega(a-b)}}{\omega(b-a)}$
Bernouli	$P(x) = \begin{cases} 1-p & x=0 \\ p & x=1 \end{cases}$ $0 < p < 1$	p	$p(1-p)$	$1-p+pe^{j\omega}$
Binomial	$P(x) = \binom{n}{x} p^x q^{n-x}$ $0 < p < 1,\ x = 0, 1, \ldots, n$ $n = 1, 2, 3, \ldots$	p	$p(1-p)$	$(1-p+pe^{j\omega})^n$
Hypergeometric	$P(x) = \frac{\binom{R}{x}\binom{N-R}{n-x}}{\binom{R}{x}}$ $N = 1, 2, 3, \ldots \ R = 0, 1, \ldots, N,$ $n = 0, 1, \ldots, N \ x = 1, \ldots, n$	nR/N	$\frac{R}{N}(1-\frac{R}{N})$ $\times n\frac{N-n}{N-1}$	Polynomial in $e^{j\omega}$
Poisson	$P(n) = \frac{\lambda^n}{n!}e^{-\lambda}$ $\lambda > 0, \quad n = 0, 1, 2, \ldots$	λ	λ	$e^{\lambda e^{j\omega} - \lambda}$
Exponential	$p(x) = \lambda e^{-\lambda x} \quad \lambda > 0,\ x > 0$	λ^{-1}	λ^{-2}	$\frac{\lambda}{\lambda - j\omega}$
Gamma	$p(x) = \frac{\lambda^\nu x^{\nu-1}}{4\Gamma(\nu)} e^{-\lambda x}$ $\lambda > 0, \quad \nu > 0, \quad x > 0$	ν/λ	ν/λ^2	$\left(\frac{\lambda}{\lambda - j\omega}\right)$
Rayleigh	$\frac{x}{\sigma^2} e^{-x^2/2\sigma^2}$ $\sigma > 0, \quad x > 0$	$\sigma\sqrt{\pi/2}$	$\sigma^2(2-\pi/2)$	$(1 + j\sigma\omega\sqrt{\pi/2})(1+$ $erf(j\sigma\omega))e^{-\sigma^2\omega^2/2}$
Inverse Gaussian	$p(x) = \frac{a}{\sqrt{2\pi}} x^{-3/2} e^{-a^2/2x}$ $a > 0, \quad x > 0$	N/A	N/A	$e^{-a\sqrt{2\omega}e^{-j\pi/4}}$
Bilateral Exponential	$p(x) = \frac{\lambda}{2} e^{-\lambda\|x\|} \ \lambda > 0$ $-\infty < x < \infty$	0	$2\lambda^{-2}$	$\frac{\lambda^2}{\lambda^2 + \omega^2}$
Beta	$p(x) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} x^{\alpha-1}(1-x)^{\beta-1}$ $\alpha > 0, \quad \beta > 0, \quad 0 < x < 1$	$\frac{\alpha}{\alpha+\beta}$	$\frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}$	$\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)}$ $\sum_{n=0}^{\infty} \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha+\beta+n)} \frac{(j\omega)^n}{k!}$
Cauchy	$p(x) = \frac{\alpha}{\pi(x^2+\alpha^2)} \quad x\alpha > 0$ $-\infty < x < \infty$	N/A	N/A	$e^{-\alpha\|\omega\|}$

9.5 Greek Alphabet

A,	α	alpha	I,	ι	iota	Σ,	σ	sigma
B,	β	beta	K,	κ	kappa	T,	τ	tau
Γ,	γ	gamma	Λ,	λ	lambda	Υ,	υ	upsilon
Δ,	δ	delta	M,	μ	mu	Φ,	ϕ	phi
E,	ϵ,	epsilon	V,	ν	nu	C,	χ	chi
Z,	ζ	zeta	Ξ,	ξ	xi	Ψ,	ψ	psi
N,	η	eta	Π,	π	pi	Ω,	ω	omega
Θ,	θ	theta	P,	ρ	rho			

9.6 Fundamental Physical Constants

The following table lists a few useful constants and their units.

Constant	Symbol	Value	Units
pi	π	3.141592653589793238	
natural number	e	2.718281828459045235	
Euler's constant	γ	0.5772157	
golden ratio	ϕ	1.618034	
speed of light*	c	2.99792458×10^{8}	m/s
atomic mass unit	u	$1.6605655 \times 10^{-27}$	kg
Avogadro's constant	N_A	6.0231×10^{23}	mol^{-1}
electric field constant*	ϵ_0	$8.854187818 \times 10^{-23}$	JK^{-1}
electron volt	eV	$1.602176634 \times 10^{-19}$	J (exact)
permeability constant*	μ_0	$4\pi \times 10^{-7}$	H/m (exactly)
Impedence of free space	Z	$376.7304 = \mu_0 c$	Ω
gravitational constant	g	6.6732×10^{-11}	Nm^2/kg^2
Boltzmann's constant	k	1.30622×10^{-23}	J K^{-1}
Faraday constant	F	96485.33212	C mol^{-1}
electron charge	e	1.60219×10^{-19}	C
Plank's constant	h	6.62620×10^{-34}	Js
reduced Plank's constant	$\hbar$	$h/2\pi$	
molar gass constant	R	8.314462618	$\text{J mole}^{-1}\ \text{K}^{-1}$
Ideal gas constant	R_0	8.31434×10^{-3}	$\text{J kmole}^{-1}/\text{K}$
Earth gravity acceleration	g	9.80665	m/s^2
Standard atmosphere	atm	101325	N/m^2 (exactly)
Electron rest mass	M_e	9.109558×10^{-31}	kg
Proton rest mass	M_p	1.672614×10^{-27}	kg
Neutron rest mass	M_n	1.67492×10^{-27}	kg

* in a vacuum

9.7 SI and MKS Units

Name	Symbol	Units	Use
ampere	A	*basic unit*	*electrical current*
candela	cd	*basic unit*	*luminous intensity*
coulomb	C	A s	electric charge
electron volt*	eV		fundamental electric charge
farad	F	A s/V	capacitance
gauss*	G		magnetic field strength
henry	H	V s/A	inductance
hertz	hz	s^{-1}	frequency
joule	J	N m	energy
kelvin	K	*basic unit*	*temperature*
kilogram	kg	*basic unit*	*mass*
lumin	lm	cd sr	amount of light
lux	lx	lm/m^2	illumination density
meter	m	*basic unit*	*length, distance*
mole	mol	*basic unit*	*atomic count*
newton	N	kg m/s	force
ohm	Ω	V/A	electrical resistence
pascal	P	N/m^2	pressure
radian	rad	*basic unit*	*angle*
second	s	*basic unit*	*time, period*
steradian	sr	*basic unit*	*solid angle*
telsa	T	Wb/m^2	magnetic flux density
volt	V	W/A	electic potential, voltage, electromotive force
watts	W	J/s	power
wave number	k	m^{-2}	spatial frequency
weber	Wb	V s	magnetic flux

* Not an MKS unit

9.8 Miscellaneous

9.8.1 Other Geophysical Constants

Constant	Symbol	Value	Units
Astronomical Unit	AU	1.496×10^8	km
Lightyear	ly	9.4605×10^{12}	km
Parsec	pc	3.0856×10^{13}	km
Equitorial radius of the Earth		6.37816×10^3	km
Polar radius of the Earth		6.35677×10^3	km
Average radius of the Earth	R_E	6.37102×10^3	km
Mass of the Earth	M_E	5.9758×10^{24}	kg
Oblatness of the Earth	f	0.003393	(none)

9.8.2 Standard Unit Multiples

The following table presents standard names and abreviations for unit multiples.

Name	Symbol	Factor
atto	a	10^{-18}
femto	f	10^{-15}
pico	p	10^{-12}
nano	n	10^{-9}
micro	μ	10^{-6}
milli	m	10^{-3}
centi	c	10^{-2}
deci	d	10^{-1}
kilo	k	10^{3}
mega	M	10^{6}
giga	G	10^{9}
tera	T	10^{15}
peta	P	10^{18}

9.8.3 Roots of Polynomial Equations

Quadratic: $ax^2 + bx + c = 0$

$$x = \frac{1}{2a}\left(-b \pm \sqrt{b^2 - 4ac}\right)$$

When a, b, and c are real the following cases occur:
1. $b^2 - 4ac > 0$: two real roots.
2. $b^2 - 4ac = 0$: one real root.
2. $b^2 - 4ac < 0$: two complex roots.

Cubic: $y^3 + py^2 + qy + r = 0$

Substitute $y = x - p/3$ to obtain $x^3 + ax + b = 0$ where

$$\begin{aligned} a &= (3q - p^2)/3 \\ b &= (2p^3 - 9pq + 27r)/27 \end{aligned}$$

Solutions for x are then given by,

$$x = a + b, \quad \pm\sqrt{-3}\frac{a-b}{2} + \frac{a+b}{2}.$$

When p, q, and r are real the following cases occur:
1. $27b^2 + 4a^3 > 0$: one real root, two conjugate imaginary roots.
2. $27b^2 + 4a^3 = 0$: three real roots, at least two will be equal.
3. $27b^2 + 4a^3 < 0$: three real, unequal roots.

9.8.4 Combinations and Permutations

The number of *combinations* of m distinct things taken n at a time is:

$$\binom{m}{n} = \frac{m!}{n!(m-n)!}$$

where, by definition,

$$\binom{m}{0} = 1.$$

The number of *permutations* of m distinct things taken n at a time is $m!/(m-n)!$. A useful approximation for the factorial function is $n! \approx e^{-n}n^n\sqrt{2\pi n}$. Note that $n! = \Gamma(n+1)$ and $x\Gamma(x) = \Gamma(x+1)$ with

$$\Gamma(x) = \int_0^\infty t^{x-1}e^{-t}dt.$$

Note that $\Gamma(1/2) = \sqrt{\pi}$.

9.8.5 Spheres and Circles

Let r be the radius, then:

Parameter	Formula
perimeter of a circle	$2\pi r$
area of a circle	πr^2
surface area of a sphere	$4\pi r^2$
volume of a sphere	$\frac{4}{3}\pi r^3$

9.8.6 Ellipse

Let a and b be the semi-major and semi-minor radii, respectively The area of an ellipse is πab. Eccentricity E of ellipse is defined as

$$E = \frac{\sqrt{a^2 - b^2}}{a}$$

The perimeter p of an ellipse is given by

$$p = 4a\int_0^{\pi/2} \sqrt{1 - e^2\sin^2\theta}\, d\theta$$

$$p = 2a\pi\left[1 - \left(\frac{1}{2}\right)^2 E^2 - \left(\frac{1\cdot 3}{2\cdot 4}\right)^2 \frac{E^4}{3} - \left(\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\right)^2 \frac{E^6}{5}\cdots\right]$$

9.8.7 Matrix Inversion

The inverse of 2×2 and 3×3 matrices are

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}^{-1} = \frac{1}{a_{12}a_{23}a_{31} - a_{13}a_{22}a_{31} + a_{13}a_{21}a_{32} - a_{12}a_{21}a_{33} + a_{11}a_{22}a_{33}} \cdot \begin{bmatrix} a_{22}a_{33} - a_{23}a_{32} & a_{13}a_{32} - a_{12}a_{33} & a_{12}a_{23} - a_{13}a_{22} \\ a_{23}a_{31} - a_{21}a_{33} & a_{11}a_{33} - a_{13}a_{31} & a_{13}a_{21} - a_{11}a_{23} \\ a_{21}a_{32} - a_{22}a_{31} & a_{12}a_{31} - a_{11}a_{32} & a_{11}a_{22} - a_{12}a_{21} \end{bmatrix}$$

More generally, let the matrix A be partitioned into submatrices B, C, D, and D,

$$A = \begin{bmatrix} B & C \\ D & E \end{bmatrix}.$$

Then A^{-1} can be expressed as (assuming inverses exist),

$$A^{-1} = \begin{bmatrix} W & X \\ Y & Z \end{bmatrix}$$

where

$$\begin{aligned} W &= (B - CE^{-1}D)^{-1} \\ Z &= (E - DB^{-1}C)^{-1} \\ X &= -B^{-1}CZ \\ Y &= -E^{-1}DW. \end{aligned}$$

9.8.8 Matrix Pseudoinverse

For a $m \times n$ matrix A, the Moore-Penrose pseudoinverse $A^\dagger$ is defined as

$$A^\dagger = \left(A^T A\right)^{-1} A^T$$

which provides a least-square solution to

$$Ax = b \quad \widehat{x} = A^\dagger b$$

where T denotes transpose and $\widehat{x}$ is an estimate of x. The rank of $A^\dagger$ is the minimum of m and n.

9.8.9 Vector Arithmetic

Let a and b be scalars and A, B, and C be vectors of compatible dimensions with elements A_i, B_i, and C_i, respectively. Then,

$$\begin{aligned}
A \cdot B &= A_1B_1 + A_2B_2 + A_3B_3 \\
A \times B &= \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \end{vmatrix} \\
\nabla^2 A &= \nabla(\nabla \cdot A) - \nabla \times (\nabla \times A) \\
A \times B &= -B \times A \\
A \times (B \times C) &= (A \cdot C)B - (A \cdot B)C \\
A \cdot (B \times C) &= B \cdot (C \times A) \ = \ C \cdot (A \times B) \\
A \cdot B &= B \cdot A \\
\nabla(ab) &= a\nabla b + b\nabla a \\
\nabla \cdot (aA) &= a\nabla \cdot A + A \cdot \nabla a \\
\nabla \times (aA) &= a(\nabla \times A) - A \times \nabla a \\
\nabla \cdot (A + B) &= \nabla \cdot A + \nabla \cdot B \\
\nabla \times (A + B) &= \nabla \times A + \nabla \times B \\
\nabla \times (A \times B) &= A(\nabla \cdot B) + (B \cdot \nabla)A - B(\nabla \cdot A) - (A \cdot \nabla)B \\
\nabla \cdot (A \times B) &= B \cdot (\nabla \times A) - A \cdot (\nabla \times B) \\
\nabla \cdot (\nabla \times A) &= 0 \\
\nabla \times \nabla a &= 0
\end{aligned}$$

The angle θ between the vectors A and B is

$$\theta = \cos^{-1}\left(\frac{A \cdot B}{|A|\,|B|}\right).$$

The projection of the vector B onto A is

$$\frac{A \cdot B}{|A|}\, A.$$

9.8.10 Coordinate Systems and Transformations

Coordinate System Transformations

To:	From: Cartesian	Spherical	Cylindrical
Cartesian	(x, y, z)	$x = r\sin\theta\cos\phi$ $y = r\sin\theta\sin\phi$ $z = r\cos\theta$	$x = \rho\cos\phi$ $y = \rho\sin\phi$ $z = z$
Spherical	$r = \sqrt{x^2 + y^2 + z^2}$ $\theta = \cos^{-1}(z/r)$ $\phi = \tan^{-1}(y/x)$	(r, ϕ, θ)	$r = \sqrt{\rho^2 + z^2}$ $\theta = \cos^{-1}(z/r)$ $\phi = \phi$
Cylindrical	$\rho = \sqrt{x^2 + y^2}$ $\phi = \tan^{-1}(y/x)$ $z = z$	$\rho = r\sin\theta$ $\phi = \phi$ $z = r\cos\theta$	(ρ, ϕ, z)

Note for polar coordinates, use cylindrical with $z = 0$ or spherical with $\theta = \pi/2$.

9.8.11 Partial Derivatives in Various Coordinate Systems

In the following, symbols with hats are unit vectors in the direction of the indicated coordinate.

Divergence ($\nabla \cdot$ F)

Given a vector function $\boldsymbol{F}$ the divergence $\nabla \cdot \boldsymbol{F}$ is:
Cartesian: $\boldsymbol{F} = (F_x(x, y, z), F_y(x, y, z), F_z(z, y, z))$

$$\nabla \cdot \boldsymbol{F} = \frac{\partial}{\partial x}F_x + \frac{\partial}{\partial y}F_y + \frac{\partial}{\partial z}F_z$$

Cylindrical: $\boldsymbol{F} = (F_\rho(\rho, \phi, z), F_\rho(\rho, \phi, z), F_z(\rho, \phi, z))$

$$\nabla \cdot \boldsymbol{F} = \frac{1}{\rho}\frac{\partial}{\partial \rho}(\rho F_\rho) + \frac{1}{\rho}\frac{\partial}{\partial \phi}F_\phi + \frac{\partial}{\partial z}F_z$$

Spherical: $\boldsymbol{F} = (F_r(r, \theta, \phi), F_\theta(r, \theta, \phi), F_\phi(r, \theta, \phi))$

$$\nabla \cdot \boldsymbol{F} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 F_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}(F_\phi \sin\theta) + \frac{1}{r\sin\theta}\frac{\partial}{\partial \phi}F_\phi$$

Gradient ($\nabla \cdot S$)

Given a scalar function S the gradient ∇S is:
Cartesian: $S(x, y, z)$

$$\nabla S = \frac{\partial}{\partial x}S\hat{x} + \frac{\partial}{\partial y}S\hat{y} + \frac{\partial}{\partial z}S\hat{z}$$

Cylindrical: $S(\rho, \phi, z)$

$$\nabla S = \frac{1}{\rho}\frac{\partial}{\partial \rho}S\hat{\rho}) + \frac{1}{\rho}\frac{\partial}{\partial \phi}S\hat{\phi} + \frac{\partial}{\partial z}S\hat{z}$$

Spherical: $S(r, \theta, \phi)$

$$\nabla S = \frac{\partial}{\partial r}S\hat{r} + \frac{1}{r}\frac{\partial}{\partial \theta}S\hat{\theta} + \frac{1}{r\sin\theta}\frac{\partial}{\partial \phi}S\hat{\theta}$$

Curl ($\nabla \times$ F)

Given a vector function $\boldsymbol{F}$ the curl $\nabla \times \boldsymbol{F}$ is:
Cartesian: $\boldsymbol{F} = (F_x(x, y, z), F_y(x, y, z), F_z(z, y, z))$

$$\nabla \times \boldsymbol{F} = \left(\frac{\partial}{\partial y}F_z - \frac{\partial}{\partial z}F_y\right)\hat{x} + \left(\frac{\partial}{\partial z}F_x - \frac{\partial}{\partial x}F_z\right)\hat{y} + \left(\frac{\partial}{\partial x}F_y - \frac{\partial}{\partial y}F_x\right)\hat{z}$$

Cylindrical: $\boldsymbol{F} = (F_\rho(\rho, \phi, z), F_\rho(\rho, \phi, z), F_z(\rho, \phi, z))$

$$\nabla \times \boldsymbol{F} = \left(\frac{1}{\rho}\frac{\partial}{\partial \phi}F_z - \frac{\partial}{\partial z}F_\phi\right)\hat{\rho} + \left(\frac{\partial}{\partial z}F_\rho - \frac{\partial}{\partial \rho}F_z\right)\hat{\phi} + \frac{1}{\rho}\left(\frac{\partial}{\partial \phi}(\rho F_\phi) - \frac{\partial}{\partial \phi}F_\rho\right)\hat{z}$$

Spherical: $\boldsymbol{F} = (F_r(r, \theta, \phi), F_\theta(r, \theta, \phi), F_\phi(r, \theta, \phi))$

$$\nabla \times \boldsymbol{F} = \frac{1}{r\sin\theta}\left(\frac{\partial}{\partial \theta}(F_\phi \sin\theta) - \frac{\partial}{\partial \phi}F_\theta\right)\hat{r} + \frac{1}{r}\left(\frac{1}{\sin\theta}\frac{\partial}{\partial \phi}F_r - \frac{\partial}{\partial \phi}(rF_\phi)\right)\hat{\theta}$$
$$+\frac{1}{r}\left(\frac{\partial}{\partial r}(rF_\theta) - \frac{\partial}{\partial \theta}F_r\right)\hat{\phi}$$

Laplacian ($\nabla^2 \cdot S$)

Given a scalar function S the Laplacian $\nabla^2 S$ is:
Cartesian: $S(x, y, z)$

$$\nabla^2 S = \frac{\partial^2}{\partial x^2}S + \frac{\partial^2}{\partial y^2}S + \frac{\partial^2}{\partial z^2}S$$

Cylindrical: $S(\rho, \phi, z)$

$$\nabla^2 S = \frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho\frac{\partial}{\partial \rho}S\right) + \frac{1}{\rho^2}\frac{\partial^2}{\partial \phi^2}S + \frac{\partial^2}{\partial z^2}S$$

Spherical: $S(r, \theta, \phi)$

$$\nabla^2 S = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}S\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial}{\partial \theta}S\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial \phi^2}S$$

Differential Elements

The differential length dl, differential area ds and differential volume dv are,

	dl	ds	dv
Cartesian	$\hat{x}dx + \hat{y}dy + \hat{z}dz$	$\hat{x}dydz + \hat{y}dxdz + \hat{z}dxdy$	$dxdydz$
Cylindrical	$\hat{\rho}d\rho + \hat{\phi}\rho d\phi + \hat{z}dz$	$\hat{\rho}\rho d\phi dz + \hat{\phi}d\rho dz + \hat{z}\rho d\rho d\phi$	$\rho d\rho d\phi d\theta$
Spherical	$\hat{r}dr + \hat{\theta}rd\theta + \hat{\phi}r\sin\theta d\phi$	$\hat{r}r^2\sin\theta d\theta d\phi + \hat{\theta}r\sin\theta drd\phi + \hat{\phi}rdrd\theta$	$r^2\sin\theta drd\theta d\phi$

Vector Integration

Let dl be the differential length, ds be the differential area, and dv be the differential volume with scalar and vector functions, a and A, respectively. Then,

$$\begin{aligned}
\int_{\text{volume}} \nabla \cdot A dv &= \oint_{\text{surface}} A \cdot ds \\
\int_{\text{volume}} \nabla A dv &= -\oint_{\text{surface}} A \times ds \\
\int_{\text{volume}} \nabla a dv &= \oint_{\text{surface}} a ds \\
\int_{\text{surface}} (\nabla \times A) ds &= \oint_{\text{line}} A \cdot dl \\
\int_{\text{surface}} A_n \times \nabla a ds &= \oint_{\text{line}} a dl
\end{aligned}$$

9.8.12 Some Useful Integrals

$$\int xe^{ax}dx = \frac{1}{a^2}e^{ax}(ax-1)$$
$$\int xe^{-x^2}dx = -\frac{1}{2}e^{-x^2}$$
$$\int \frac{e^{ax}}{b+ce^{ax}}dx = \frac{1}{ac}\log(b+ce^{ax})$$
$$\int e^{ax}\sin(bx)dx = \frac{e^{ax}[a\sin bx - b\cos bx]}{a^2+b^2}$$
$$\int e^{ax}\cos(bx)dx = \frac{e^{ax}[a\cos bx + b\sin bx]}{a^2+b^2}$$
$$\int xe^{ax}\sin(bx)dx = \frac{xe^{ax}[a\sin bx - b\cos bx]}{a^2+b^2}$$
$$\int xe^{ax}\cos(bx)dx = \frac{xe^{ax}[a\sin bx + b\cos bx]}{a^2+b^2}$$

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