# A DIFFERENTIAL FORMS APPROACH TO ELECTROMAGNETICS IN ANISOTROPIC MEDIA

A Dissertation Submitted to the Department of Electrical and Computer Engineering Brigham Young University

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## A DIFFERENTIAL FORMS APPROACH TO ELECTROMAGNETICS IN ANISOTROPIC MEDIA

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#### ABSTRACT

The behavior of electromagnetic fields in an inhomogeneous, anisotropic medium can be characterized by a tensor Green function for the electric field. In this dissertation, a new formalism for tensor Green functions using the calculus of differential forms is proposed. Using this formalism, the scalar Green function for isotropic media is generalized to an anisotropic, inhomogeneous medium. An integral equation is obtained relating this simpler Green function to the desired Green function for the electric field, generalizing the standard technique for construction of the Green function for the isotropic case from the scalar Green function. This treatment also leads to a new integral equation for the electric field which is a direct generalization of a standard free space result. For the special case of a biaxial medium, a paraxial approximation for the Green function is used to obtain the Gaussian beam solutions. A straightforward analysis breaks down for beams propagating along two singular directions, or optical axes, so these directions are investigated specially. The associated phenomenon of internal conical refraction is known to yield a circular intensity pattern with a dark ring in its center; this analysis predicts the appearence of additional dark rings in the pattern.

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This dissertation by Karl F. Warnick is accepted in its present form by the Department of Electrical and Computer Engineering of Brigham Young University as satisfying the dissertation requirement for the degree of Doctor of Philosophy.

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## **DEDICATION**

To my wife Shauna for her support and faith in me.

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#### Chapter 1

## **INTRODUCTION**

Electromagnetic fields interact with the materials in which they exist. On the atomic scale, the interactions between fields and particles can be extremely complex, but on a macroscopic scale, the influence of a medium on fields can be modelled by modifying the constitutive relations between the electric and magnetic field intensity and the associated flux densities. These constitutive relations, together with Maxwell's laws, govern the propagation of fields in materials. A medium for which the relationship between field and flux density depends on the direction of field intensity is an anisotropic medium. If the constitutive relations depend on position, then the medium is inhomogeneous. A bianisotropic medium is one in which electric and magnetic fields are coupled by the constitutive relations. In this dissertation I consider the behavior of electromagnetic fields in an anisotropic, inhomogeneous medium. Bianisotropic, nonlinear, and spatially dispersive media are not considered. The term complex media is often used to denote the class of materials of the most general type, but "complex media" or "general media" will be used here to denote the limited category under consideration. The most general constitutive relation to be treated are possibly position dependent, linear relationships of the form  $D_i = \epsilon_{ij}(\mathbf{r})E_j$  and  $B_i = \mu_{ij}(\mathbf{r})H_j$ , where  $\epsilon_{ij}$  is the permittivity tensor and  $\mu_{ij}$  is the permeability tensor. In the general derivations of Chapters 3 and 4, the only restriction placed on the constitutive tensors is that they must be non-singular. Special cases are treated thereafter. Chapters 5 and 6 deal with biaxial materials, which are homogeneous, magnetically isotropic, and have a diagonalizable permittivity tensor with three unique eigenvalues. I consider only time-harmonic  $(e^{-i\omega t})$  fields, so that effects due to temporal dispersion are neglected. Although many of the general results given in this dissertation are coordinate-free, I employ rectangular coordinates almost exclusively when dealing with expressions in component form.

Numerous types of materials fall into the class treated in this dissertation. Anisotropic media are employed in electromagnetic devices for modulation and control of signals, especially those materials for which the anisotropy can be influenced by application of a static or slowly varying electric field and devices which employ polarization-dependent effects to control microwave and optical signals. Anisotropic effects of the ionosphere must be studied in order to understand the behavior of radio waves for which transmission is affected by this region of the atmosphere. Problems involving inhomogeneous media are ubiquitous, and range from investigations of interaction between a biological object and a radiating antenna to statistical analysis of effects on signal propagation due to random fluctuation of atmospheric properties. Inhomogeneous media arise in a variety of remote sensing applications, and their effects must be quantified in order to effectively evaluate and interpret data obtained by detection of signals radiated or scattered by natural or artificial materials. Inhomogeneous materials such as graded index fibers are often employed in optical systems. Problems for which the medium could be considered both inhomogeneous and anisotropic include the scattering problem for bounded anisotropic materials of various shapes, layered anisotropic media, or anisotropic coatings.

Methods for analysis of fields in complex media are manifold. Possible approaches include computational algorithms for solving differential and integral equations as well as analytical approaches specialized to particular problems. The particular method to be extended and applied here is the theory of the tensor Green function for the electric field. Maxwell's laws can be solved for an arbitrary source configuration and a specified boundary condition if an appropriate tensor Green function is available. The tensor Green function essentially represents the electric field produced by an infinitesimal current source of arbitrary orientation and location. If this Green function is known, then the fields due to a given source can be obtained by direct integration, so that the Green function can be thought of as completely characterizing the electromagnetic properties of a particular medium.

For a general medium, a closed form representation of the Green function has not been obtained. For an inhomogeneous medium, the problem of determining the Green function is especially difficult, since information about the variation of the medium over the entire region of interest must be incorporated into the Green function. Even for a biaxial medium, the Green function can only be given in closed form asymptotically. The present understanding of Green functions for complex media is far from complete, and the research reported in this dissertation is intended to advance this area of electromagnetic field theory.

Chapter 2 is devoted to a study of previous work on Green functions for complex media and an introduction to the primary tool used in this dissertation, the calculus of differential forms. The power of differential forms as a tool for electromagnetics is the foundation of the results of this dissertation. As outlined briefly in Chap. 2 and in detail in Appendix B and Ref. [1], the calculus of differential forms offers both algebraic and geometrical advantages over traditional vector analysis. With differential forms, many vector identities and theorems are reduced to simple, algebraic properties. This makes differential forms ideal in searching for new theoretical approaches, since manipulations are often more transparent and less tedious than they would be if the usual notation were employed. Differential forms also allow field quantities and the laws they obey to be visualized in an intuitive manner. This is valuable in research since problems can be understood and solved first visually and then mathematically. The geometrical representation for electromagnetic boundary conditions given in Chap. 2, for example, is naturally related to the mathematical expression derived in Ref. [2] and Appendix A.

In order to employ the calculus of differential forms to treat the theory of electromagnetic Green functions, I represent the tensor Green function as a double differential form, rather than as a dyadic. The utility of double forms for the case of free space has been demonstrated in Ref. [3], where it is shown that differential forms make key expressions more concise and easier to apply in some respects than their dyadic formulations. In order to treat a general medium, I construct in Chap. 3 Hodge star operators from the permittivity and permeability tensors. The new formalism arising from the use of these star operators yields two benefits: first, the same few fundamental theorems and algebraic properties of the calculus of differential forms which are used to treat electromagnetics in free space can be employed for complex media with only minor modification. Second, expressions extend in a more obvious way to the inhomogeneous, anisotropic case, facilitating the generalization of free space results to complex media. Some results generalize to a complex medium simply by reinterpreting the star operators which are already present in the expressions.

After using this formalism to define the Green form for the electric field, I recover known results for the electric field in terms of the Green form, impressed sources, and boundary values of the fields due to sources external to the region of interest. Unlike previous treatments, this derivation follows the pattern of the standard, formal theory of Green functions by obtaining key results from a generalization of Green's theorem. With the derivation cast into this form, the origins of symmetry and self–adjointness properties of the Green form and the associated differential operator become clear. The treatment also elucidates the role of boundary conditions in determining the properties of the Green function and the associated differential operator.

For a homogeneous, isotropic medium, the tensor Green function can be constructed from a simpler Green function associated with the scalar Helmholtz equation. Similar techniques have been sought for anisotropic media with limited success in certain special cases, as will be reviewed in Chap. 2. The primary intent of Chapter 3 is to generalize this type of construction. While I do not obtain a closed form solution for the Green function, the treatment does yield a result that is a rather direct generalization of the free space method. Using the wave operator of the calculus of differential forms, I generalize to a complex medium the scalar Helmholtz equation and the associated free space Green function. The associated Green function is a double form rather than a scalar quantity, but is still simpler than the Green form for the electric field. This Helmholtz Green form can be obtained analytically for an unbounded, homogeneous, anisotropic medium. For an isotropic medium, it reduces to a double form with the usual scalar Green function as the diagonal component. Following introduction of the Helmholtz Green form, the central result of this work is derived: a relationship between the Helmholtz Green form and the Green form for the electric field. In free space, the Green form for the electric field can be expressed in terms of the scalar Green function and its derivatives. For a complex medium, this relationship becomes an integral equation. Although the integral equation does not reduce directly to the free space expression, the two constructions are very similar in form. The work contained in Chap. 3 has been reported in Ref. [4].

Chapter 4 treats in more detail an integral equation for the electric field in terms of the Helmholtz Green form of the previous chapter. The equivalence of this integral equation with a standard result for the electric field due to sources in an isotropic, homogeneous medium is demonstrated. The isotropic expression is manipulated into a form that generalizes directly to the case of a complex medium. I contrast this integral equation with the usual integral equation method for complex media, and discuss cases where the present approach may have advantage over the usual method. I also give a principal value interpretation for integrals involving derivatives of the Helmholtz Green form, which is required in order to implement the integral equation numerically.

Following these general considerations, I specialize to the case of a biaxial medium. Chapter 5 treats the propagation of Gaussian beams in biaxial media. I give the beam solutions and parameters in terms of the direction of propagation and the permittivity of the medium. There are two singular directions, or optical axes, for which the results of Chap. 5 break down. Narrow beams in these directions spread into a hollow cone. This phenomenon is known as internal conical refraction. In Chap. 6, I give a special analysis of beams for these directions, obtaining an expression for field intensities that yields new features of internal conical refraction not discerned by previous theories. The material in this chapter is also reported in Ref. [5]. It has long been known that the characteristic, annular intensity pattern produced by internal conical refraction of a narrow beam exhibits in its center a fine, dark ring. This dark ring has been observed and explained theoretically. The analysis presented here indicates the existence of secondary dark rings concentric to the primary dark ring on the interior of the intensity pattern. For a biaxial medium, these secondary fringes have apparently not been observed or predicted, although similar dark rings have been reported for an optically active crystal [6]. I give quantitative results for the field intensity at various parameter values and specify the parameter regime for which this effect should appear.

In summary, the contributions of this dissertation to electromagnetic field theory in general and the study of electromagnetic propagation in complex media are:

- A new formalism based on the Hodge star operator for electromagnetic Green functions in complex media;
- A generalization of the Helmholtz equation to anisotropic, inhomogeneous media, the definition of the associated Green form, and the solution for the Helmholtz Green form for the case of a homogeneous, anisotropic medium;
- An integral equation relating the Green form for the electric field to the Helmholtz Green form which generalizes the standard construction for the free space Green function;
- A new electric field integral equation with kernel related to the Helmholtz Green form which is a direct generalization of a standard free space result;
- A generalization of the free space Stratton–Chu formula to complex media;
- Explicit representation of Gaussian beam solutions for generic propagation directions in a biaxial medium;
- A precise analysis of internal conical refraction of a Gaussian beam with wave direction along an optical axes of a biaxial medium, and the prediction of new structure in the associated intensity pattern.

The results of this research include not only the solution of specific problems, but also a new theoretical approach to the theory of anisotropic, inhomogeneous media, with the definition of the Helmholtz Green form and integral equation relating the Green form for the electric field to the Helmholtz Green form. There are many special cases for which approximate or exact methods of solutions for this integral equation might be sought. Numerical methods based on this equation might also be developed. In the conclusion to this dissertation, several of the more obvious avenues for further work are noted.

#### Chapter 2

## BACKGROUND

The problem of electromagnetic propagation in anisotropic media has a long history [7], and some aspects of the theory are well understood. The plane wave solutions in a biaxial medium are known [8], as are the plane wave solutions in a general homogeneous medium [9]. The existence and uniqueness of solutions to the general problem of Maxwell's laws with specified sources and boundary condition and arbitrary constitutive relations have been treated in the mathematics literature [10, 11]. For types of fields other than the plane waves in a complex medium, however, exact solutions are difficult to obtain. Since wave solutions for an arbitrary source can be determined from the tensor Green function by direct integration, much of the work on fields in complex media has been directed towards the search for exact or asymptotic representations of the Green function. In this chapter, I will review past contributions to the theory of tensor Green function for complex media. I will then give a brief introduction to the calculus of differential forms and its applications in electromagnetics, since this is the primary tool used in this dissertation to treat Green functions.

#### 2.1 Green Function Methods for Complex Media

The primary intent of the research effort reported in this dissertation is to develop a new theoretical method for the treatment of propagation in complex media which will lead to an exact representation of the Green function for such materials. For a uniaxial medium, the tensor Green function has been given in closed form [14]. For a biaxial medium, the near field limit of the tensor Green function is known [15], as well as the far field limit for generic directions in the medium [16]. The singular behavior of fields in the medium propagating in certain directions necessitates a more careful analysis, but for the far field limit, this analysis has been completed [17]. A series solution for the Green form of a biaxial medium has also been found in terms of vector wave functions [18], but an exact, closed form solution is not known. For an inhomogeneous medium, the problem of finding the Green function is even more difficult than for a homogeneous, anisotropic medium. Closed form representations must be sought using methods specialized to particular types of inhomogeneity, although general numerical methods for determination of the Green function for inhomogeneous media are available [19].

As noted in the introduction, the main result of this dissertation is a generalization of the free space construction of the tensor Green function for the electric field in terms of a simpler Green function which can be obtained exactly. This type of representation has been sought by other researchers, with success for certain limits or types of materials. Weiglhofer gives the tensor Green function for a uniaxial medium in closed form in terms of scalar Green functions [14]. The Green function for an isotropic, inhomogeneous medium has also been represented in terms of two simpler quantities satisfying coupled partial differential equations [20]. The coupled equations can be solved for media varying only in one dimension and in the limit of a weakly inhomogeneous medium. The far field limit of the Green function for a biaxial medium can be expressed in terms of scalar quantities which have the same form as the free space scalar Green function [16]. For a general complex medium, however, a representation of this type for the tensor Green function has not been has not been obtained in the past.

#### 2.2 Present Approach

The theory developed in the following chapters relies on a new notation for electromagnetics in complex media based on the calculus of differential forms. The tensor Green function is represented as a double differential form, or Green form, as done for free space by Thirring [12] in the spacetime representation and Ref. [3] in the 3 + 1 representation. This approach can be extended to the case of a complex medium by embedding the permittivity and permeability tensors into the Hodge star operator, rather than employing them directly as tensor quantities. The use of the Hodge star operator to characterize material properties was suggested in passing by Bamberg and Sternberg [13]. This new notation allows the the identities and theorems of the calculus of differential forms which are used for electromagnetics in free space to be applied to the theory of complex media.

The calculus of differential forms is widely used in various fields of physics and mathematics, and its advantages over traditional vector and tensor methods have been noted by many authors. In Sec. 2.3, I give a brief outline of some areas in which differential forms are used, and then survey in more detail applications within the field of electromagnetics. In order to provide background for the following chapters, Sec. 2.4 gives a brief introduction to the quantities, operators, and key theorems of the calculus of differential forms, including the exterior product, exterior derivative, the generalized Stokes theorem, and the interior product. Maxwell's laws, the free space constitutive relations, and boundary conditions are represented using differential forms. These and other topics are treated in greater detail in the Appendices.

#### 2.3 Survey of the Calculus of Differential Forms

A differential form is a quantity that can be integrated, including differentials. More precisely, a differential form is a fully covariant, fully antisymmetric tensor [21, 22]. The calculus of differential forms was developed from the exterior algebra of Grassman by Cartan, Poincaré and others in the early 1900's, and like vector analysis is a self–contained subset of tensor analysis.

Differential forms are used regularly in fields of physics such as general relativity [23], quantum field theory [24], thermodynamics [13], and mechanics [25]. A section on differential forms is commonplace in mathematical physics texts [26, 27]. Differential forms have been applied to control theory by Hermann [28] and others. Systems of differential forms are currently a prominent method in nonlinear control theory, and differential forms methods are used to search for symmetries of nonlinear differential equations [29]. In applied electromagnetics, however, vector analysis was already entrenched by the time the calculus of differential forms became widely known. In spite of this, a number of authors have employed differential forms to treat various aspects of EM theory.

Aside from early papers in which Maxwell's laws were originally written using differential forms, the general relativity text by Misner, Thorne and Wheeler [23] is one of the first works to emphasize the use of differential forms in electromagnetics. Since the focus of the work is gravitation, applications of EM theory are not treated. Burke [30]

treats a range of mathematical physics topics. The chapter on electromagnetics gives an elegant formulation of electromagnetic boundary conditions. Bamberg and Sternberg [13] also develop various topics of mathematical physics. Maxwell's equations appear as the continuous limit of the laws of circuit theory expressed using discrete differential forms.

Other works include that of Ingarden and Jamiołkowksi [31], an electrodynamics text using a mix of vectors and differential forms, and the advanced electrodynamics text by Parrott [32]. Thirring [12] is a classical field theory text which treats general relativity in addition to electromagnetics, but certain applied topics such as waveguides are included. Thirring represents an electromagnetic Green function as a double differential form, and derives a result analogous to that of Sec. 3.2 for free space in the spacetime formulation. Flanders [25] is a standard reference on the mathematical aspects and applications of differential forms.

Deschamp was among the first to suggest the use of differential forms in engineering. His article [33] considers briefly several applications, such as Huygen's principle and reciprocity. The papers [34], [35], [36], [37], [38], [39], [40] are essentially similar to previous treatments, with additional applications such as Čerenkov radiation [36] or the Hertz potentials [39]. Reference [41] advocates a variational technique derived using differential forms for numerical solution of electromagnetics problems, and Ref. [42] suggests a numerical method for computation of fields in elastic, conducting media based on a method for the discretization of electromagnetic field and source differential forms. Sasaki and Kasai [43] review the algebraic topology of the differential forms representing the electromagnetic field. Burke also gives an interesting discussion of electromagnetics using twisted differential forms [44], so that parity invariance is explicit and a "right–hand rule" is not required. The papers [45, 46] employ differential forms to treat the relativistic rotation of a charged particle in an electromagnetic field.

More recent work includes that of Kotiuga, who uses differential forms to solve the problem of making cuts for magnetic scalar potentials in multiply connected regions [47] and to provide a metric–independent functional for the variational solution of electromagnetic inverse problems. Baldwin has investigated the use of Clebsch potentials to represent field quantities [48] and classified the principle linearly polarized electromagnetic waves [49]. References [1] and [50] describe the intuitive geometrical viewpoint which differential forms provide for the principles of electromagnetics; this material is included in Appendix B.

#### 2.4 Introduction to the Calculus of Differential Forms

This section provides a brief, elementary introduction to the calculus of differential forms. A more comprehensive treatment also at an elementary level can be found in Ref. [1] and Appendix B. The references noted above offer more advanced and rigorous discussions.

#### 2.4.1 Degree of a Differential Form; Exterior Product

The calculus of differential forms is the calculus of quantities that can be integrated. The degree of a form is the dimension of the region over which it is integrated. For the remainder of this section we restrict attention to differential forms in three dimensions, so that there exist 0-forms, 1-forms, 2-forms, and 3-forms. 0-forms are simply functions, and are "integrated" by evaluation at a point.



Figure 2.1: (a) The 1-form dx. (b) The 2-form dy dz. Tubes in the z direction are formed by the superposition of the surfaces of dy and the surfaces of dz. (c) The 3-form dx dy dz, with three sets of surfaces that create boxes.

A 1-form is integrated over a path, and under the condition given in Sec. 2.4.4 can be represented graphically by surfaces, as in Fig. 2.1a. The surfaces of a 1-form have an associated orientation, represented by a choice of one of the two normals of each surface. The general 1-form a(x, y, z) dx + b(x, y, z) dy + c(x, y, z) dz is said to be *dual* to the vector field  $a(x, y, z)\hat{\mathbf{x}} + b(x, y, z)\hat{\mathbf{y}} + c(x, y, z)\hat{\mathbf{z}}$  in the euclidean metric. The integral of a 1-form over a path is the number of surfaces pierced by the path, taking into account the orientation of the surfaces and the direction of integration.

2-forms are integrated over surfaces. The general 2-form  $a(x, y, z) dy \wedge dz + b(x, y, z) dz \wedge dx + c(x, y, z) dx \wedge dy$  is dual to the vector field  $a(x, y, z)\hat{\mathbf{x}} + b(x, y, z)\hat{\mathbf{y}} + c(x, y, z)\hat{\mathbf{z}}$  in the euclidean metric. The wedge  $\wedge$  between differentials represents the exterior product, which for 1-forms is anticommutative, so that  $dx \wedge dy = -dy \wedge dx$  and  $dx \wedge dx = 0$ . Wedges are often dropped for compactness. The exterior product is the antisymmetrized tensor product, so that  $A \wedge B = A \otimes B - A \otimes B$ , where A and B are rank one tensors.

Graphically, 2-forms can be represented by tubes (Fig. 2.1b). As the coefficients of a 2-form increase, the tubes become denser. The tubes are oriented in the direction of the associated dual vector. The integral of a 2-form over a surface is equal to the number of tubes passing through the surface, where each tube contributes a positive or negative value depending on the relative orientations of the tube and the surface.

A 3-form is a volume element, represented by boxes (Fig. 2.1c). The greater the magnitude of a 3-form's coefficient, the smaller and more closely spaced are the boxes. The integral of a 3-form over a volume is the number of boxes inside the volume, where each box is weighted by the sign of the 3-form. The general 3-form q(x, y, z) dx dy dz is dual to its coefficient q(x, y, z). Forms of degree greater than three vanish by the anticommutativity of the exterior product.

The electric and magnetic field intensities E and H are 1-forms; their surfaces represent equipotentials if the fields are conservative. The electric and magnetic flux densities D and B are 2-forms, as well as the electric current density J. The electric charge density  $\rho$  is a 3-form with coefficient equal to the usual charge density scalar. Each box of the 3-form represents a certain amount of charge. Each of these differential forms is dual to the corresponding vector or scalar quantity.

#### 2.4.2 Maxwell's Laws in Integral Form

Using the differential forms for field and source quantities defined above, Maxwell's laws can be written as

$$\oint_{P} E = -\frac{d}{dt} \int_{A} B$$

$$\oint_{P} H = \frac{d}{dt} \int_{A} D + \int_{A} J$$

$$\oint_{S} D = \int_{V} \rho$$

$$\oint_{S} B = 0$$
(2.1)

where A is a surface bounded by a path P and V is a volume bounded by a surface S. As discussed in Appendix B, the units of E and H are V and A, D and B have units C and Wb, and the sources J and  $\rho$  have units of A and C, since the differentials in these forms are considered to have units of length.



Figure 2.2: (a) Gauss's law: boxes of electric charge produce tubes of electric flux. (b) Ampere's law: tubes of current produce magnetic field surfaces. (c) Tubes of D are perpendicular to surfaces of E, since  $D = \epsilon_0 \star E$ .

Gauss's law for the electric field shows that a closed surface containing a certain number of boxes of the electric charge density 3-form must be pierced by a like number of tubes of the electric flux density 2-form. Thus, it has the geometrical interpretation that tubes of electric flux emanate from boxes of electric charge, as illustrated by Fig. 2.2a. Gauss's law for the magnetic field requires that tubes of magnetic flux density never end.

Ampere's law shows that in a similar way tubes of electric current or time– varying electric flux produce magnetic field intensity surfaces (Fig. 2.2b). Each closed path through which tubes of electric current or time–varying electric flux pass must pierce the same number of surfaces of the magnetic field intensity 1-form. With vectors, Ampere's law and the curl operator are not as intuitive as Gauss's law and the divergence, but with differential forms, Ampere's and Faraday's laws obtain a geometrical meaning that is as simple as that of Gauss's law. These graphical representations are discussed more fully in Appendix B.

#### **2.4.3** The Hodge Star Operator and the Constitutive Relations

The Hodge star operator is a set of isomorphisms between p-forms and (n-p)forms, where n is the dimension of the underlying space. The star operator is dependent on
a metric, as will be discussed further in Chap. 3. In  $R^3$  with the euclidean metric,

$$\star dx = dy \, dz, \ \star dy = dz \, dx, \ \star dz = dx \, dy$$

and  $\star 1 = dx dy dz$ . Also,  $\star \star = 1$ , so that the euclidean star operator is its own inverse. The constitutive relations in free space are  $D = \epsilon_0 \star E$  and  $B = \mu_0 \star H$ , where  $\epsilon_0$  is the permittivity and  $\mu_0$  is the permeability of the vacuum. Graphically, tubes of flux are perpendicular to surfaces of field intensity, as depicted in Fig. 2.2c. For the anisotropic star operator which will be used in Chap. 3, tubes of flux are skew to surfaces of field intensity.

#### 2.4.4 The Exterior Derivative and Maxwell's Laws in Point Form

The exterior derivative can be written formally as

$$d = \left(\frac{\partial}{\partial x}dx + \frac{\partial}{\partial y}dy + \frac{\partial}{\partial z}dz\right) \wedge$$
(2.2)

and acts like the vector gradient operator on 0-forms, the curl on 1-forms, and the divergence on 2-forms. In practice, the computational rule for the exterior derivative can be stated simply as "take the partial derivative of a quantity by each coordinate and add the corresponding differential from the left." The exterior derivative of f dx, for example, is  $f_x dx dx + f_y dy dx + f_z dz dx = f_z dz dx - f_y dx dy$ , where subscripts represent partial derivatives. The exterior derivative of a product of differential forms expands as  $d(\alpha \wedge \beta) = (d\alpha) \wedge \beta + (-1)^p \alpha \wedge d\beta$ , where p is the degree of  $\alpha$ .

The exterior derivative allows a condition to be given for the existence of the geometrical representation of 1-forms given in Sec. 2.4.1. In R<sup>3</sup>, the solution to Pfaff's problem [51] shows this type of geometrical representation exists for a 1-form  $\omega$  provided that  $\omega \wedge d\omega = 0$ . If  $\omega \wedge d\omega \neq 0$ , then there exist coordinates for which  $\omega = du + v dw$ , so that  $\omega$  is the sum of two differential forms which can be represented individually by surfaces. In R<sup>2</sup> each 1-form can be represented graphically by lines.

An arbitrary, smooth 2-form in  $\mathbb{R}^3$  can be written locally in the form  $f \, dg \wedge dh$ [22], so that in the coordinates (f, g, h) the 2-form can be represented as tubes of  $dg \wedge dh$ scaled by f.

The generalized Stokes theorem is

$$\int_{M} d\omega = \int_{\partial M} \omega \tag{2.3}$$

where  $\omega$  is a *p*-form and *M* is a (p + 1)-dimensional region with boundary  $\partial M$ . This relationship is equivalent to the fundamental theorem of calculus if  $\omega$  is a 0-form, the vector Stokes theorem if  $\omega$  is a 1-form, and the divergence theorem if  $\omega$  is a 2-form.

Using the exterior derivative and the generalized Stokes theorem, Maxwell's laws can be written as

$$dE = -\frac{\partial B}{\partial t} \tag{2.4a}$$

$$dH = \frac{\partial D}{\partial t} + J$$
 (2.4b)

$$dD = \rho \tag{2.4c}$$

$$dB = 0. \tag{2.4d}$$

The physical nature of each field quantity is no longer contained in the type of derivative operator acting on it, but rather is expressed solely by the degree of the differential form representing the quantity. Derivations are often more straightforward with differential forms than they are when vectors are employed, since the algebraic properties of the exterior derivative and other operators are largely independent of the degrees of the forms involved and so a small number of theorems and identities suffice for most manipulations.

#### 2.4.5 The Interior Product and Boundary Conditions

The interior product of a vector and a differential form is the usual tensor contraction. With the use of a metric, the interior product of differential forms can be defined, by raising the tensor indices of the first form to make it a vector or multivector and then contracting it with the left–most index or indices of the second form. In this dissertation, the same symbol  $\ \ \$  will be used for both the contraction of a vector and a form as well as the metric–dependent interior product of two differential forms.

In the the euclidean metric, the interior product of differential forms reduces to a few simple relationships. For pairs of 1-forms,  $dx \, \lrcorner \, dx = dy \, \lrcorner \, dy = dz \, \lrcorner \, dz = 1$  and all other combinations vanish. For the interior product of a 1-form and a 2-form,

$$dz \, \lrcorner \, (dz \land dx) = - \, dy \, \lrcorner \, (dx \land dy) = \, dx$$
$$dx \, \lrcorner \, (dx \land dy) = - \, dz \, \lrcorner \, (dy \land dz) = \, dy$$
$$dy \, \lrcorner \, (dy \land dz) = - \, dx \, \lrcorner \, (dz \land dx) = \, dz$$

and  $dx \, \lrcorner (dy \land dz) = dy \, \lrcorner (dz \land dx) = dz \, \lrcorner (dx \land dy) = 0$ . The interior product can also be written in terms of the star operator:

$$a \, \lrcorner b = \star (\star b \land a). \tag{2.5}$$

Graphically, the interior product removes the surfaces of the first form from those of the second.

Boundary conditions on the electromagnetic field can be written using the operator  $n \, \lrcorner n \land$ , where *n* is the normalized exterior derivative df/|df| of a function f(x, y, z)which vanishes along a boundary surface. In Appendix A and Ref. [2] it is shown that

$$n \, \lrcorner \left( n \land \left[ E_1 - E_2 \right] \right) = 0$$

$$n \sqcup (n \land [H_1 - H_2]) = J_s$$
$$n \sqcup (n \land [D_1 - D_2]) = \rho_s$$
$$n \sqcup (n \land [B_1 - B_2]) = 0$$

where  $J_s$  is the surface current density 1-form,  $\rho_s$  is the surface charge density 2-form, and the subscript 1 represents field values above (f > 0) and the subscript 2 below (f < 0) the boundary.

These expressions for boundary conditions have a simple geometric interpretation. The discontinuity  $H_1 - H_2$ , for example, is a 1-form with surfaces that intersect the boundary along the lines of the 1-form  $J_s$  (Fig. 2.3a). Thus, restricted to the boundary,  $H_1 - H_2$  is equal to  $J_s$ . The operator  $n \, \lrcorner n \land$  simply removes the component of the field which has zero restriction to the boundary. In the expression for  $J_s$ , the exterior product  $n \land (H_1 - H_2)$  creates tubes with sides perpendicular to the boundary (Fig. 2.3b). The interior product  $n \, \lrcorner (n \land [H_1 - H_2])$  removes the surfaces that were added by the exterior product, as shown in Fig. 2.3c. The total effect of the operator  $n \, \lrcorner n \land$  is to select the component of  $H_1 - H_2$  with surfaces perpendicular to the boundary.



Figure 2.3: (a) The field discontinuity  $H_1 - H_2$ , which has the same intersection with the boundary as  $J_s$ . (b) The exterior product  $n \wedge [H_1 - H_2]$ yields tubes running along the boundary, with sides perpendicular to the boundary. (c) The interior product with n removes the surfaces parallel to the boundary, leaving surfaces that intersect the boundary along the lines representing the 1-form  $J_s$ .

Unlike other differential forms of electromagnetics,  $J_s$  is not dual to the usual surface current density vector  $J_s$ . The expression for current through a path P is

$$I = \int_{P} \mathbf{J}_{\mathbf{s}} \cdot (\hat{\mathbf{n}} \times d\hat{\mathbf{s}})$$
(2.6)

where  $\hat{\mathbf{n}}$  is a surface normal and  $\hat{\mathbf{s}}$  is tangent to the path. Using the 1-form  $J_s$ , this simplifies to

$$I = \int_P J_s \tag{2.7}$$

which is the obvious definition for a surface current quantity.

#### 2.4.6 Integration by Pullback

Integrals of differential forms can be evaluated in a straightforward manner using the method of pullback. A vector field must be converted to a differential form before it can be integrated. This accounts for the presence of an inner product in the path or surface integral of vector field. The method of pullback is more natural, since neither a metric nor a differential vector is required to evaluate an integral of a form. To integrate a 1-form  $\omega$  over a path P parameterized as (u(s), v(s), w(s)) in an arbitrary coordinate system (u, v, w), the coordinates u, v and w in the arguments of the coefficients as well as the differentials of  $\omega$ are replaced with u(s), v(s) and w(s). Jacobian factors enter automatically when the exterior derivatives du(s), dv(s), and dw(s) are computed. The result of the pullback operation is a new 1-form which can be written as g(s) ds. This 1-form is the pullback of  $\omega$  to the path P, and is integrated over the limits of the parameter s of the path. If  $\omega$  is the 1-form f(x, y, z) dx, for example, then the integral of  $\omega$  over the path P is

$$\begin{split} \int_{P} \omega &= \int_{P} f(x, y, z) \, dx \\ &= \int_{b}^{a} f(u(s), v(s), w(s)) \, du(s) \\ &= \int_{b}^{a} f(u(s), v(s), w(s)) \frac{\partial u}{\partial s} \, ds \end{split}$$

Integration of a 2-form over a surface proceeds similarly, except that two parameters s and t are necessary and the final integrand is a 2-form in  $ds \wedge dt$ .

#### 2.5 Summary

In this chapter, I have given a survey of various results contained in the literature on the theory of electromagnetic Green functions which relate to the work reported in this dissertation. I have also outlined the calculus of differential forms, since this will be the primary tool to be employed in the following chapters. Chapter 3, which constitutes the core of this dissertation, begins by generalizing the euclidean star operator of Sec. 2.4.3 to an asymmetric, complex metric, so that the star operator can be used to express the constitutive relations for materials with arbitrary permeability and permittivity tensors. This formalism enables other operators and theorems of the calculus of differential forms to be used in obtaining the key result of this research: a new representation for the Green function for the electric field for anisotropic, inhomogeneous media.

#### Chapter 3

# GREEN FORMS FOR ANISOTROPIC, INHOMOGENEOUS MEDIA

The goal of this chapter is to represent the tensor Green function for a complex medium in terms of a simpler Green function which can be obtained exactly, generalizing the standard construction method for the tensor Green function in free space. The material given here is also contained in Ref. [4].

In order to derive this result, the tensor Green function is represented as a double differential form. This method is employed to treat the special case of free space in Ref. [3]. For the general case, in Sec. 3.1 material properties as characterized by the permittivity and permeability tensors are embedded into the Hodge star operator. The usual definition of the Hodge star operator must be modified for material tensors with negative or complex determinants. In addition, the metric tensor from which the Hodge star operator is defined is by definition symmetric. In order to employ the star operator to characterize media with nonsymmetric material tensors  $\epsilon_{ij}$  and  $\mu_{ij}$ , the definition of the Hodge star operator must be extended in a formal manner. Fortunately, this new operator retains many of the same properties as the usual, symmetric Hodge star operator, as demonstrated in Sec. 3.1. As far as the derivations of this chapter are concerned, the primary difference between the symmetric and nonsymmetric star operators is that the nonsymmetric star operator is not proportional to its own inverse.

Following these preparatory derivations, in Sec. 3.2 I define the Green form for the electric field and recover known results [52] for the electric field in terms of the Green form and current sources. The derivation presented in this chapter is analogous to the standard treatment of the general theory of Green functions [27]. As a result, the origins of conventions used in the definition of the Green form and symmetry and self–adjointness properties of the Green form and the associated partial differential operator become clear. The reformulation of the tensor Green function as a double differential form and the use of the Hodge star operator to express constitutive relations lead to a natural generalization of the Helmholtz equation to anisotropic media. Unlike the Green form for the electric field, the Green form for this generalized Helmholtz equation can be found exactly for an important class of media, those which are homogeneous and anisotropic. This class is quite general, since it includes biaxial media, lossy media, and nonreciprocal media such as gyrotropic plasma. For an isotropic medium, the Helmholtz Green form essentially reduces to the usual scalar Green function.

In Sec. 3.3 the main result of this chapter is given: an integral equation relating the Green form for the electric field to the Helmholtz Green form. The kernel of this integral equation consists of second order partial derivatives of the Helmholtz Green form. The expression obtained in this chapter does not reduce directly to the usual result for free space, since the usual result gives the electric field directly from the sources, while the expression given here remains an integral equation even for free space. The integral equation and the free space relationship, however, are very similar in form and have a clear connection. The correspondence between the treatment of this chapter and standard free space results is treated in Chap. 4.

By specializing to a homogeneous medium, this integral equation can be transformed into the wavevector representation, leading to known expressions for the Fresnel equation and the Fourier transform of the Green form for the electric field. The Neumann series solution for the integral equation in the wavevector representation can be resummed, yielding another type of representation for the Green form.

#### 3.1 The Hodge Star Operator for a Complex Medium

In Sec. 2.4.3, the Hodge star operator was used to express the free space constitutive relations. It was noted there that the Hodge star operator depends on a metric. If this metric is related in the proper way to the permittivity and permeability tensors, the free space constitutive relations of Sec. 2.4.3 can be generalized to the case of a complex medium. In order to treat media which have nonsymmetric permittivity or permeability tensors, however, the standard definition of the Hodge star operator must be extended in a formal manner. The standard definition must also be modified if the determinants of the material tensors are not real and positive, as can occur if a medium is lossy. After making the necessary generalizations, I determine the inverse of the star operator, prove the theorem  $\nu \wedge \star \lambda = \star^{-1} \nu \wedge \lambda$  for *p*-forms  $\nu$  and  $\lambda$ , and define the Laplace–de Rham or wave operator.

The most commonly used definition for the Hodge star operator is that given by Flanders [25] and Bamberg and Sternberg [13],

$$\lambda \wedge \nu = (\star \lambda, \nu)\sigma \tag{3.1}$$

where  $\nu$  is a *p*-form,  $\lambda$  is an (n-p)-form,  $\sigma$  is the volume element  $\star 1 \equiv \sqrt{|g|} dx^1 \wedge \cdots \wedge dx^n$ and (,) denotes the inner product of *p*-forms induced by the metric tensor  $g_{ij}$ . Thirring [12] gives an alternate definition,

$$\star \lambda = \lambda \, \lrcorner \, \sigma \tag{3.2}$$

where  $\Box$  denotes the interior product on differential forms induced by the metric  $g_{ij}$ . These two definitions can be shown to be equivalent using the relationship  $\lambda \wedge \star \omega = (\lambda \sqcup \omega)\sigma$ where  $\lambda$  and  $\omega$  are *p*-forms. Let  $\nu = \star \omega$ . Then by making use of  $\star \star \omega = (-1)^{p(n-p)+s}\omega$ , Eq. (3.1) becomes

$$\lambda \wedge \nu = (-1)^{p(n-p)+s} (\lambda \, \lrcorner \, \star \nu) \sigma. \tag{3.3}$$

Thirring shows that  $(-1)^{p(n-p)+s}(\lambda \sqcup \star \nu)$  is equal to the inner product of the *p*-forms  $\star \lambda$ and  $\nu$ , so that this expression reduces to the definition (3.1). The text [23] on p. 97 also defines a duality between *p*-forms and (n - p)-vectors. If the metric is used to lower the indices of the (n - p)-vector, the resulting (n - p)-form is equivalent to that obtained by applying the star operator to the original *p*-form (note that the tensor  $\varepsilon$  used in Ref. [23] contains a factor of  $\sqrt{|g|}$ ).

For the purposes of this chapter, an explicit definition of the star operator in terms of a metric is most useful. For a simple p-form,

$$\star dx^{i_1} \wedge \ldots \wedge dx^{i_p} = g^{i_1 j_1} \ldots g^{i_p j_p} \varepsilon_{j_1 \ldots j_n} \frac{\sqrt{|g|}}{(n-p)!} dx^{j_{p+1}} \wedge \cdots \wedge dx^{j_n}$$
(3.4)

where  $\varepsilon$  is the Levi-Civita tensor, g is the determinant of the metric tensor, n is the dimension of space, and  $g^{ij}$  is the inverse metric. The derivation of this expression from (3.2) is given as an exercise in Ref. [12]. For the euclidean metric  $\delta_{ij}$ , we recover the result given in Sec. 2.4.3, that  $\star dx = dy dz$ ,  $\star dy = dz dx$ ,  $\star dz = dx dy$ , and  $\star 1 = dx dy dz$ .

For symmetric, positive definite permittivity and permeability tensors, I define  $\star_e$  using (3.4) with the inverse metric  $g^{ij} = \epsilon_{ji}/(\det \epsilon_{ij})$  and  $\star_h$  with  $g^{ij} = \mu_{ji}/(\det \mu_{ij})$ . The constitutive relations can then be written as

$$D = \star_e E \tag{3.5a}$$

$$B = \star_h H. \tag{3.5b}$$

Since a metric tensor is by definition symmetric, the definition (3.4) produces a true Hodge star operator only if  $g^{ij} = g^{ji}$ . By employing the expression formally with a nonsymmetric  $g^{ij}$ , however, an operator is obtained which retains many of the useful properties of the Hodge star operator. This allows the treatment given in this chapter to apply to nonreciprocal media, for which the material tensors are nonsymmetric.

Due to the presence of the absolute value in the factor  $\sqrt{|g|}$  of Eq. (3.4), the definition of the star operator must also be modified if the determinants of the material tensors are not positive and real. I therefore define the star operator employed in this chapter according to

$$\star dx^{i_1} \wedge \ldots \wedge dx^{i_p} = g^{i_1 j_1} \ldots g^{i_p j_p} \varepsilon_{j_1 \ldots j_n} \frac{\sqrt{g}}{(n-p)!} dx^{j_{p+1}} \wedge \cdots \wedge dx^{j_n}.$$
(3.6)

Using this definition, the constitutive relations (3.5) are valid for an anisotropic, inhomogeneous, nonbianisotropic, and linear medium. The operator obtained using the modified definition, as well as its formal extension to a nonsymmetric tensor  $g^{ij}$ , is still referred to as a star operator and given the same symbol  $\star$  throughout this dissertation.

In rectangular coordinates, from Eq. (3.6) the star operator  $\star_e$  acts on an arbitrary 1-form in the obvious way, so that

$$\star_{e}(E_{1} dx + E_{2} dy + E_{3} dz) = (\epsilon_{11}E_{1} + \epsilon_{12}E_{2} + \epsilon_{13}E_{3}) dy dz + (\epsilon_{21}E_{1} + \epsilon_{22}E_{2} + \epsilon_{23}E_{3}) dz dx + (\epsilon_{31}E_{1} + \epsilon_{32}E_{2} + \epsilon_{33}E_{3}) dx dy$$
(3.7)

where wedges between differentials are omitted. If the star operator  $\star_e$  is applied to a 2-form,

$$\star_{e}(D_{1} dy dz + D_{2} dz dx + D_{3} dx dy) = (\epsilon^{11}D_{1} + \epsilon^{21}D_{2} + \epsilon^{31}D_{3}) dx + (\epsilon^{12}D_{1} + \epsilon^{22}D_{2} + \epsilon^{32}D_{3}) dy + (\epsilon^{13}D_{1} + \epsilon^{23}D_{2} + \epsilon^{33}D_{3}) dz$$
(3.8)

where the  $\epsilon^{ij}$  are components of  $\epsilon^{-1}$ . On 1-forms and 3-forms,

$$\star_e 1 = (\det \epsilon_{ij}) \, dx \, dy \, dz. \tag{3.9}$$

The magnetic star operator  $\star_h$  behaves similarly.

For media with symmetric permeability and permittivity tensors,  $\star_e = \star_e^{-1}$  and  $\star_h = \star_h^{-1}$ . As can be seen by inspection of Eq. (3.8), in the nonsymmetric case the star operator is no longer equal to its own inverse. As will be shown below, the inverse of the star operator must in general be defined using (3.6) with  $g^{ij}$  replaced by its transpose  $g^{ji}$ . I give this transposed star operator the symbol  $\tilde{\star}$ . The inverse star operator  $\star_e^{-1} = \tilde{\star}_e$  is thus obtained from (3.6) with  $g^{ij} = \epsilon_{ij}/(\det \epsilon_{ij})$  and  $\star_h^{-1} = \tilde{\star}_h$  with  $g^{ij} = \mu_{ij}/(\det \mu_{ij})$ .

I now prove that  $\tilde{\star}$  is proportional to  $\star^{-1}$  for a nonsymmetric  $g^{ij}$  by demonstrating the result for a simple *p*-form. The general case follows by linearity of the star operator. Applying the definition (3.6) and using the shorthand notation  $dx^{i_1} \wedge \cdots \wedge dx^{i_p} = dx^{i_1 \dots i_p}$ ,

$$\tilde{\star} \star dx^{i_1 \dots i_p} = \frac{g}{p!(n-p)!} g^{k_{p+1}j_{p+1}} \cdots g^{k_n j_n} g^{i_1 j_1} \cdots g^{i_p j_p} \varepsilon_{k_{p+1} \dots k_n k_1 \dots k_p} \varepsilon_{j_1 \dots j_n} dx^{k_1 \dots k_p}.$$

Raising the indices of  $\varepsilon_{j_1...j_n}$  and using the expression  $(1/g)\delta_{j_1...j_n}^{i_1...i_n} = \varepsilon_{j_1...j_n}\varepsilon^{i_1...i_n}$  for the permutation tensor  $\delta$  in terms of the Levi–Civita tensor gives

$$\tilde{\star} \star dx^{i_1 \dots i_p} = \frac{1}{p!(n-p)!} g^{k_{p+1}j_{p+1}} \cdots g^{k_n j_n} g_{j_{p+1}l_{p+1}} \cdots g_{j_n l_n} \delta^{i_1 \dots i_p l_{p+1} \dots l_n}_{k_{p+1} \dots k_n k_1 \dots k_p} dx^{k_1 \dots k_p}.$$

Using the definition  $g_{ij}g^{jk} = \delta_i^k$  of the inverse metric  $g^{jk}$  and permuting the indices  $k_1 \dots k_n$ , this becomes

$$\tilde{\star} \star dx^{i_1 \dots i_p} = \frac{1}{p!(n-p)!} \delta_{l_{p+1}}^{k_{p+1}} \cdots \delta_{l_n}^{k_n} (-1)^{p(n-p)} \delta_{k_1 \dots k_n}^{i_1 \dots i_p l_{p+1} \dots l_n} dx^{k_1 \dots k_p}.$$

Summing the indices  $l_{p+1} \dots l_n$  gives

$$\tilde{\star} \star dx^{i_1 \dots i_p} = \frac{1}{p!(n-p)!} (-1)^{p(n-p)} \delta^{i_1 \dots i_p k_{p+1} \dots k_n}_{k_1 \dots k_n} dx^{k_1 \dots k_p}.$$

Due to the antisymmetry of the permutation tensor, the quantity  $\delta_{k_1...k_n}^{i_1...i_pk_{p+1}...k_n}$  vanishes if  $k_n$  is equal to any of  $i_1, \ldots, i_p$ . Thus, there are n-p possible values for  $k_n$  which contribute to the summation over  $k_n$ , so that the summation introduces a factor of (n-p). If  $k_{n-1}$  is equal to  $k_n$  or any of  $i_1, \ldots, i_p$ , the quantity also vanishes, so  $k_{n-1}$  has n-p-1 possible values and the  $k_{n-1}$  summation yields a factor of (n-p-1). By similar reasoning, after summing over  $k_{p+1}$  through  $k_{n-2}$ , we have

$$\tilde{\star} \star dx^{i_1 \dots i_p} = \frac{1}{p!(n-p)!} (-1)^{p(n-p)} (n-p)! \delta^{1_1 \dots i_p}_{k_1 \dots k_p} dx^{k_1 \dots k_p}.$$

Since both the permutation tensor and  $dx^{k_1...k_p}$  are antisymmetric in the indices  $k_1...k_p$ , the right-hand side consists of p! copies of  $(1/p!)(-1)^{p(n-p)} dx^{k_1...k_p}$ , so that we have finally

$$\tilde{\star} \star dx^{i_1 \dots i_p} = (-1)^{p(n-p)} dx^{i_1 \dots i_p}.$$

This proves the relationship

$$\star^{-1} = (-1)^{p(n-p)} \tilde{\star} \tag{3.10}$$

so that in  $\mathbb{R}^3$ ,  $\star^{-1} = \tilde{\star}$ .

The identity  $\nu \wedge \star \lambda = \star^{-1} \nu \wedge \lambda$  for *p*-forms  $\nu$  and  $\lambda$  is required for the derivations in Sections 3.2 and 3.3. Thirring [12] proves the result for a symmetric star operator; I generalize to the nonsymmetric case. The proof is given for simple forms and extends to the general case by linearity. By the definition (3.6) of the star operator,

$$dx^{i_1...i_p} \wedge \star dx^{j_1...j_p} = \frac{\sqrt{g}}{(n-p)!} g^{j_1k_1} \cdots g^{j_pk_p} \varepsilon_{k_1...k_n} \, dx^{i_1...i_pk_{p+1}...k_n}.$$
(3.11)

By rearranging the differentials using the antisymmetry of the exterior product of 1-forms,

$$dx^{i_1\dots i_p} \wedge \star dx^{j_1\dots j_p} = \frac{\sqrt{g}}{(n-p)!} g^{j_1k_1} \cdots g^{j_pk_p} \varepsilon_{k_1\dots k_n} \varepsilon^{i_1\dots i_pk_{p+1}\dots k_n} dx^{1\dots n}.$$

This can be rewritten using the permutation tensor  $\delta$ ,

$$dx^{i_1\dots i_p} \wedge \star dx^{j_1\dots j_p} = \frac{\sqrt{g}}{(n-p)!} g^{j_1k_1} \cdots g^{j_pk_p} \frac{(n-p)!}{g} \delta^{i_1\dots i_p}_{k_1\dots k_p} dx^{1\dots n}.$$

Using an explicit representation [27] for  $\delta$ , Eq. (3.11) becomes

$$dx^{i_1...i_p} \wedge \star dx^{j_1...j_p} = \frac{1}{\sqrt{g}} \sum_{\pi} g^{j_1 i_{\pi(1)}} \cdots g^{j_p i_{\pi(p)}} \operatorname{sgn}(\pi) \, dx^{1...n}$$
(3.12)
where  $\pi$  represents a permutation of p objects. By rearranging the order of the  $g^{j_k i_{\pi(k)}}$ , this can be transformed into

$$dx^{i_1...i_p} \wedge \star dx^{j_1...j_p} = \frac{1}{\sqrt{g}} \sum_{\pi} g^{j_{\pi(1)}i_1} \cdots g^{j_{\pi(p)}i_p} \operatorname{sgn}(\pi) dx^{1...n}$$

since for each permutation  $\pi$ , the inverse permutation is also included in the summation. Reversing the steps leading to Eq. (3.12), we find that

$$dx^{i_1...i_p} \wedge \star dx^{j_1...j_p} = \frac{\sqrt{g}}{(n-p)!} g^{k_1 i_1...k_p i_p} \varepsilon_{k_1...k_n} dx^{j_1...j_p k_{p+1}...k_n} \\ = \frac{\sqrt{g}}{(n-p)!} g^{k_1 i_1...k_p i_p} \varepsilon_{k_1...k_n} (-1)^{p(n-p)} dx^{k_{p+1}...k_n j_1...j_p}.$$

Using Eq. (3.10) together with the definition of  $\tilde{\star}$  shows that

$$dx^{i_1...i_p} \wedge \star dx^{j_1...j_p} = \star^{-1} dx^{i_1...i_p} \wedge dx^{j_1...j_p}.$$
(3.13)

In  $\mathbb{R}^3$  the inverse star operator in this expression can be replaced with  $\tilde{\star}$ .

Finally, I extend the definition of the Laplace–de Rham or wave operator  $\Delta$  to allow use of the nonsymmetric star operator.  $\Delta$  is a generalization of the Laplacian. Variation in sign conventions for  $\Delta$  exists in the literature; the two alternatives are found in Bamberg and Sternberg [13] and Thirring [12]. I choose Thirring's definition, since it agrees with the sign of the usual vector Laplacian. Accordingly, I define

$$\Delta \alpha = (-1)^{n(p+1)} \left[ (-1)^n \star d\tilde{\star} d + d\tilde{\star} d\star \right] \alpha \tag{3.14}$$

where  $\alpha$  is a *p*-form. This is equivalent to Thirring's definition for a positive definite metric, and for a constant metric with real eigenvalues it differs by the sign  $|g|/g = (-1)^s$ , where *s* is the signature of  $g_{ij}$ . For a constant but otherwise arbitrary tensor  $g^{ij}$ , in a particular coordinate system (3.14) reduces to

$$\Delta(\omega \, dx^{i_1 \dots i_p}) = g^{ij} \frac{\partial^2 \omega}{\partial x_i \partial x_j} \, dx^{i_1 \dots i_p} \tag{3.15}$$

which becomes the usual expression for the Laplacian in the euclidean metric  $g_{ij} = g^{ij} = \delta_{ij}$ .

## 3.2 The Green Form for the Electric Field

In this section, I define the Green form for the electric field, and derive an expression for the observed electric field due to sources and external fields in terms of the operator transpose of the Green form. I consider a linear, nonbianisotropic medium with macroscopic electromagnetic properties characterized by invertible permittivity and permeability tensors  $\epsilon_{ij}(\mathbf{r})$  and  $\mu_{ij}(\mathbf{r})$ . Maxwell's laws are

$$dE = i\omega B \tag{3.16a}$$

$$dH = -i\omega D + J \tag{3.16b}$$

$$dD = \rho \tag{3.16c}$$

$$dB = 0 \tag{3.16d}$$

where E and H are the electric and magnetic field intensity 1-forms, D and B are the electric and magnetic flux density 2-forms, J is the electric current density 2-form, and  $\rho$  is the electric charge density 3-form. The constitutive relations are  $D = \star_e E$  and  $B = \star_h H$  where the star operators  $\star_e$  and  $\star_h$  are defined in the previous section.

By applying the operator  $\star_h d\tilde{\star}_h$  to Faraday's law and making use of the constitutive relations and Ampere's law, it can be shown that the electric field *E* satisfies

$$(-\star_h d\tilde{\star}_h d + \omega^2 \star_h \star_e) E = -i\omega \star_h J \tag{3.17}$$

where the field quantities and the star operators are evaluated at the same point. The natural Green double  $1 \otimes 1$  form G for this system of partial differential equations obeys the same equation, but with  $i\omega J$  replaced by an elementary or delta function source:

$$(-\star_h d\tilde{\star}_h d + \omega^2 \star_h \star_e) G(\mathbf{r}_1, \mathbf{r}_2) = -\star_h \delta(\mathbf{r}_1 - \mathbf{r}_2) I$$
(3.18)

where I is the unit  $2 \otimes 1$  form  $dy_1 dz_1 \otimes dx_2 + dz_1 dx_1 \otimes dy_2 + dx_1 dy_1 \otimes dz_2$  and  $\otimes$  denotes the tensor product. In rectangular coordinates, the Green form G has components

$$G(\mathbf{r}_{1}, \mathbf{r}_{2}) = G_{11} dx_{1} \otimes dx_{2} + G_{12} dx_{1} \otimes dy_{2} + G_{13} dx_{1} \otimes dz_{2} + G_{21} dy_{1} \otimes dx_{2} + G_{22} dy_{1} \otimes dy_{2} + G_{23} dy_{1} \otimes dz_{2} + G_{31} dz_{1} \otimes dx_{2} + G_{32} dz_{1} \otimes dy_{2} + G_{33} dz_{1} \otimes dz_{2}.$$
(3.19)

where the coefficients  $G_{ij}$  are functions of  $\mathbf{r}_1$  and  $\mathbf{r}_2$ . The tensor product  $\otimes$  is explicitly included in these expressions to show that there is no exterior product between the differentials of the  $\mathbf{r}_1$  and  $\mathbf{r}_2$  coordinate systems.

For an isotropic medium, the definition of G becomes

$$\left(-\frac{1}{\mu}\star d\frac{1}{\mu}\star d+\frac{\omega^2\epsilon}{\mu}\right)G(\mathbf{r}_1,\mathbf{r}_2) = -\frac{1}{\mu}\delta(\mathbf{r}_1-\mathbf{r}_2)I$$
(3.20)

where  $\epsilon(\mathbf{r})$  and  $\mu(\mathbf{r})$  are the scalar permittivity and permeability,  $\star$  is the euclidean star operator, and I is the unit  $1 \otimes 1$  form  $dx_1 \otimes dx_2 + dy_1 \otimes dy_2 + dz_1 \otimes dz_2$ . The  $\star_h$  operator acting from the left on both sides of (3.18) does not affect the definition of G but is retained since  $-\star_h d\tilde{\star}_h d$  is part of the Laplace-de Rham operator to be employed in Sec. 3.3.

In Eq. (3.18) and other expressions throughout this chapter, operators act on the  $\mathbf{r}_1$  coordinates unless otherwise noted. The star operator is in general a function of position, and is evaluated at the position vector of the coordinate system corresponding to the differentials on which it operates. Although  $\mathbf{r}_1$  in the definition (3.18) represents observation coordinates and  $\mathbf{r}_2$  represents source coordinates, the standard approach to Green function theory employed in this chapter naturally leads to a reversal of the roles of the two coordinates. For reciprocal or lossless media, the symmetry relations for the Green form obtained in Sec. 3.2.2 allow the coordinates to be interchanged.

I note here an important difference between this notation for double differential forms and the usual dyadic formulation. With double forms, the coordinate system to which each differential belongs is explicitly specified. With dyadics, the unit vectors of each component are not associated with a particular coordinate system. The information contained in the coordinate dependence of the differentials is for dyadics contained in the ordering of dot products with other quantities. With double forms, the ordering of factors is not important as far as the coordinate dependence is concerned. Thus, with double differential forms the order of exterior products can be interchanged with a possible sign change depending on the degrees of the forms. The operation corresponding to the transpose of a dyadic becomes interchange of the coordinate dependence of the coefficients of a double form, since the differentials are explicitly associated with the coordinate systems of the arguments of the double form. I define the formal transpose of G to be the  $1 \otimes 1$  double form  $\tilde{G}$  satisfying

$$(-\tilde{\star}_h d\star_h d + \omega^2 \tilde{\star}_h \tilde{\star}_e) \tilde{G}(\mathbf{r}_1, \mathbf{r}_2) = -\tilde{\star}_h \delta(\mathbf{r}_1 - \mathbf{r}_2) I.$$
(3.21)

This definition for  $\tilde{G}$  differs from Chew's [52] definition for the dyadic Green function for an anisotropic, inhomogeneous medium due to the presence of the operator  $\tilde{\star}_h$  on the left– hand side of (3.21). In general,  $\tilde{G}$  is not the coordinate transpose of the double form G, although in Sec. 3.2.1 it is shown that  $\tilde{G}(\mathbf{r}_1, \mathbf{r}_2) = G(\mathbf{r}_2, \mathbf{r}_1)$  for certain types of boundary conditions.

Let L and L represent the operators on the left hand sides of Eqs. (3.18) and (3.21) respectively. In order to obtain the electric field in terms of  $\tilde{G}$ , L and L must be such that a relationship of the form

$$E_1 \wedge (\tilde{\star}_h \mathbf{L} E_2) - E_2 \wedge (\star_h \mathbf{L} E_1) = dP$$
(3.22)

holds for arbitrary  $E_1$  and  $E_2$ . This equation will lead to a generalized Green theorem, from which symmetry and self-adjointness properties of G for reciprocal and lossless media as well as the solution for the electric field in terms of sources can be conveniently obtained.

The product rule for the exterior derivative [25],  $d(\alpha \wedge \beta) = (d\alpha) \wedge \beta + (-1)^p \alpha \wedge d\beta$ , where  $\alpha$  is a *p*-form, and the relationship  $\nu \wedge \star \lambda = \tilde{\star} \nu \wedge \lambda$  for *p*-forms  $\nu$  and  $\lambda$  obtained in the previous section can be used to show that

$$d(E_1 \wedge \tilde{\star}_h dE_2 + \star_h dE_1 \wedge E_2) = d\star_h dE_1 \wedge E_2 - E_1 \wedge d\tilde{\star}_h dE_2.$$
(3.23)

Applying this identity to (3.22) and using the definitions of L and  $\tilde{L}$  yields

$$P = E_1 \wedge \tilde{\star}_h dE_2 + \star_h dE_1 \wedge E_2 \tag{3.24}$$

for the conjunct of  $E_1$  and  $E_2$ . Note that star operators cannot be moved across the exterior products in this expression since  $E_1$  and  $E_2$  do not have the same degree as  $dE_2$  and  $dE_1$ . Integrating Eq. (3.22) over a volume  $V_1$  and applying the generalized Stokes theorem

$$\int_{\partial V} \omega = \int_{V} d\omega \tag{3.25}$$

yields a generalization of Green's theorem for the operators L and  $\tilde{L}$ ,

$$\int_{V_1} E_1 \wedge (\tilde{\star}_h \mathbf{L} E_2) - \int_{V_1} E_2 \wedge (\star_h \tilde{\mathbf{L}} E_1) = \int_{\partial V_1} P$$
(3.26)

where  $\partial V_1$  denotes the boundary of  $V_1$ . This relationship shows that  $\tilde{L}$  is the formal transpose of L with respect to the inner product

$$\langle E_1, E_2 \rangle = \int_V E_1 \wedge \tilde{\star}_h E_2.$$
 (3.27)

If the surface term vanishes, then Eq. (3.26) becomes for L and  $\tilde{L}$  the definition of operator transpose with respect to the inner product (3.27). The term formal refers to the possibility that L and  $\tilde{L}$  may have different domains; the spaces of functions on which they act may, for example, satisfy different boundary conditions.

I now wish to replace  $E_1(\mathbf{r}_1)$  with  $\tilde{G}(\mathbf{r}_1, \mathbf{r}_2)$  in (3.26). Yaghjian [53] shows that for the case of an isotropic, homogeneous medium, the identity (3.26) with this substitution is not valid if  $\mathbf{r}_1 = \mathbf{r}_2$ , since  $\tilde{G}(\mathbf{r}_1, \mathbf{r}_2)$  does not have continuous and unique second derivatives at  $\mathbf{r}_1 = \mathbf{r}_2$ . This is due to ambiguity in the interpretation of the elementary source  $\delta I$ . The resulting inconsistency can be resolved by employing an appropriate principal value interpretation for volume integrals involving  $\tilde{G}$ .

Proceeding with the substitution and applying the definition (3.21) produces the generalized Huygens principle for anisotropic, inhomogeneous media,

$$E(\mathbf{r}_2) = i\omega \int_{V_1} \tilde{G}(\mathbf{r}_1, \mathbf{r}_2) \wedge J(\mathbf{r}_1) + \int_{\partial V_1} \left[ \tilde{G}(\mathbf{r}_1, \mathbf{r}_2) \wedge \tilde{\star}_h dE(\mathbf{r}_1) + \star_h d\tilde{G}(\mathbf{r}_1, \mathbf{r}_2) \wedge E(\mathbf{r}_1) \right].$$
(3.28)

This is equivalent to the dyadic result given in [52]. Note the absence of surface normal vectors in this expression. Normal components of the fields naturally do not contribute to the surface integral term of (3.28).

### 3.2.1 Boundary Conditions

In this section, I seek to determine boundary conditions on  $E_1$  and  $E_2$  and the Green forms such that the surface terms on the right-hand sides of (3.26) and (3.28) vanish. I assume here that  $\tilde{G}(\mathbf{r}_1, \mathbf{r}_2)$  as a function of  $\mathbf{r}_1$  satisfies the same boundary condition as  $E_1$ and  $G(\mathbf{r}_1, \mathbf{r}_2)$  as a function of  $\mathbf{r}_1$  satisfies the same boundary condition as  $E_2$ . If the surface contribution is zero, then replacing  $E_1(\mathbf{r}_1)$  with  $\tilde{G}(\mathbf{r}_1, \mathbf{r}_2)$  and  $E_2(\mathbf{r}_1)$  with  $G(\mathbf{r}_1, \mathbf{r}_3)$  in (3.26) shows that

$$\tilde{G}(\mathbf{r}_3, \mathbf{r}_2) = G(\mathbf{r}_2, \mathbf{r}_3). \tag{3.29}$$

Thus, for properly chosen boundary conditions,  $\tilde{G}$  is the coordinate transpose of G. In addition, the electric field as given by (3.28) will satisfy the same boundary condition as  $\tilde{G}(\mathbf{r}_1, \mathbf{r}_2)$  as a function of  $\mathbf{r}_2$ . By Eq. (3.29), the electric field will then satisfy the same boundary condition as  $G(\mathbf{r}_1, \mathbf{r}_2)$  as a function of  $\mathbf{r}_1$ . Boundary conditions for which the surface contribution vanishes include radiation, Neumann (magnetically conducting), and Dirichlet (electrically conducting). There are other boundary conditions for which the surface term of (3.28) vanishes; this section treats only the simplest types.

Suppose that  $E_2$  has the asymptotic behavior

$$\lim_{r \to \infty} r \left[ dE_2 - ik_2 \, dr \wedge E_2 \right] = 0 \tag{3.30a}$$

$$\lim_{r \to \infty} r(\star_h dr) \wedge E_2 = 0 \tag{3.30b}$$

$$\lim_{r \to \infty} r |E_2| \leq C \tag{3.30c}$$

where  $k_2$  is a bounded function of **r** and *C* is a constant. Let  $E_1$  have the same behavior but with  $k_2$  replaced with  $k_1$  in (3.30a) and  $\star_h$  replaced with  $\tilde{\star}_h$  in (3.30b). If  $\partial V$  is a sphere with radius *r*, then

$$\lim_{r \to \infty} \left| \int_{\partial V} E_1 \wedge \tilde{\star}_h \left[ dE_2 - ik_2 \, dr \wedge E_2 \right] \right| \leq \lim_{r \to \infty} 4\pi r^2 \sup_{\partial V} |E_1| \sup_{\partial V} |dE_2 - ik_2 \, dr \wedge E_2|$$

which vanishes by conditions (3.30a) and (3.30c). Using this result, we have that

$$\lim_{r \to \infty} \int_{\partial V} P = \lim_{r \to \infty} \int_{\partial V} \left[ E_1 \wedge i k_2 \tilde{\star}_h (dr \wedge E_2) + i k_1 \star_h (dr \wedge E_1) \wedge E_2 \right].$$
(3.31)

Using (2.5), the integrand can be rewritten using the interior product as

$$-ik_2 \check{\star}_h [E_1 \, \check{\lrcorner} \, (\, dr \wedge E_2)] + ik_1 \star_h [E_2 \, \lrcorner \, (\, dr \wedge E_1)]$$

where  $\ \ \,$  is the interior product induced by the metric of  $\star_h$  and  $\tilde{\ \, }$  is induced by  $\tilde{\star}_h$ . The interior products expand by (A.7) to become

$$-ik_2(E_1 \,\widetilde{\,\,}\, dr)\check{\star}_h E_2 + ik_1(E_2 \,\lrcorner\, dr)\star_h E_1 + ik_2(E_1 \,\widetilde{\,\,}\, E_2)\check{\star}_h dr - ik_1(E_2 \,\lrcorner\, E_1)\star_h dr.$$

By making use again of (2.5), this becomes

$$-ik_{2}[\check{\star}_{h}(\check{\star}_{h} dr \wedge E_{1})]\check{\star}_{h}E_{2} + ik_{1}[\star_{h}(\star_{h} dr \wedge E_{2})]\star_{h}E_{1}$$
$$+ik_{2}[\check{\star}_{h}(\check{\star}_{h}E_{2} \wedge E_{1})]\check{\star}_{h} dr - ik_{1}[\star_{h}(\star_{h}E_{1} \wedge E_{2})]\star_{h} dr.$$

The first two terms of this expression vanish in the limit when integrated over  $\partial V$  due to the conditions (3.30b) and (3.30c), by an argument similar to that used in arriving at (3.31). The second pair of terms can be written using (3.13) and (2.5) as

$$i(E_2 \,\lrcorner\, E_1)(k_2 \check{\star}_h - k_1 \star_h) \, dr$$

In order for this to vanish, we must have that the permeability tensor is symmetric. We must also have that the asymptotic forms of  $E_1$  and  $E_2$  as expressed by (3.30a) be identical. A sufficient condition for this is that the permittivity tensor be symmetric. Thus, for a medium with symmetric permeability and permittivity tensors and outgoing fields satisfying the conditions (3.30), the surface contribution to (3.26) vanishes. It remains to show for specific types of media that the fields behave according to (3.30).

For electrically conducting boundary conditions, the 1-forms  $E_1$  and  $E_2$  are oriented perpendicular to the boundary, so that if n is a coordinate normal to the boundary, then  $E_1$  and  $E_2$  are proportional to dn. The 2-forms  $E_1 \wedge \tilde{\star}_h dE_2$  and  $\star_h dE_1 \wedge E_2$  therefore must each contain a factor of dn. Since the surface integration is over all coordinates except n, the boundary term of Eq. (3.26) vanishes.

For magnetically conducting boundary conditions, the 1-form  $H_2$  is oriented perpendicular to the boundary, so that  $H_2$  must be proportional to dn. We have from Faraday's law and the constitutive relation for B that  $\tilde{\star}_h dE_2 = i\omega H_2$ , so that  $\tilde{\star}_h dE_2$  is also proportional to dn. In general,  $E_1$  need not satisfy the same boundary condition as  $E_2$  (physically,  $E_2$  satisfies Maxwell's laws with constitutive relations as given by (3.5), whereas  $E_1$  satisfies Maxwell's laws with the magnetic constitutive relation  $B = \tilde{\star}_h H$ ). If we require that  $\star_h dE_1$  be oriented perpendicular to the boundary, then this 1-form contains a factor of dn as well. For these conditions on  $E_1$  and  $E_2$ , the boundary term of Eq. (3.26) vanishes.

## 3.2.2 Symmetry and Self-Adjointness Conditions

The reaction of a field E and source J is

$$\langle E, J \rangle_R = \int_V E \wedge J$$
 (3.32)

where E is a 1-form and J is a 2-form. The derivation of Eq. (3.26) together with the relationship (3.17) show that if  $\star_h = \tilde{\star}_h$  and  $\star_e = \tilde{\star}_e$ , then

$$\langle E_2, J_1 \rangle_R - \langle E_1, J_2 \rangle_R = \int_{\partial V} (E_1 \wedge H_2 + H_1 \wedge E_2)$$
 (3.33)

where  $E_1$  and  $E_2$  here are solutions of (3.17) with sources  $J_1$  and  $J_2$  respectively. This is the Lorentz reciprocity theorem. For boundary conditions such that the right-hand side vanishes, Eq. (3.33) reduces to the definition of reciprocity,  $\langle E_2, J_1 \rangle_R = \langle E_1, J_2 \rangle_R$ . Thus, we recover the result that a medium is reciprocal if  $\star_h$  and  $\star_e$  are symmetric and the fields satisfy boundary conditions such that the surface contribution of Eq. (3.33) vanishes. Making use again of (3.17), we find from the definition of reciprocity that

$$\int_{V_1} E_1 \wedge (\star_h \mathbf{L} E_2) = \int_{V_1} E_2 \wedge (\star_h \mathbf{L} E_1)$$
(3.34)

which shows that L is symmetric with respect to the inner product (3.27). Replacing  $E_1$  with  $G(\mathbf{r}_1, \mathbf{r}_2)$  and  $E_2$  with  $G(\mathbf{r}_1, \mathbf{r}_3)$  in Eq. (3.34) gives the reciprocity relation [52]

$$G(\mathbf{r}_3, \mathbf{r}_2) = G(\mathbf{r}_2, \mathbf{r}_3) \tag{3.35}$$

for a medium with symmetric permittivity and permeability tensors and fields satisfying boundary conditions such that the surface contribution of Eq. (3.33) vanishes.

The energy imparted to the field E by the source J is

$$\langle E, J \rangle_E = \int_V E^* \wedge J.$$
 (3.36)

By slightly modifying the derivation of (3.26), one can show that if  $\star_e = \tilde{\star}_e^*$  and  $\star_h = \tilde{\star}_h^*$ , then

$$\langle E_1, J_2 \rangle_E + \langle E_2, J_1 \rangle_E^* = \int_{\partial V} (H_1^* \wedge E_2 - E_1^* \wedge H_2).$$
 (3.37)

The superscript \* on  $\tilde{\star}_e^*$  and  $\tilde{\star}_h^*$  denotes complex conjugation of the coefficients of the permittivity and permeability tensors employed in the definitions of the star operators. For a given source J and associated field E, setting  $J_1 = J_2 = J$  and  $E_1 = E_2 = E$  in (3.37) yields

$$\operatorname{Re} \langle E, J \rangle_{E} = -\operatorname{Re} \int_{\partial V} E \wedge H^{*}.$$
(3.38)

This equation represents a balance between energy imparted to the field by the source J and the power flow through the boundary of V, so that the material in the region V must be lossless (note that this expression is the real part of Poynting's theorem for a lossless medium).

If in addition the boundary conditions are assumed to be such that the righthand side of (3.37) vanishes (so that no real power flows through the boundary of V), then we have that

$$\langle E_1, J_2 \rangle_E = -\langle E_2, J_1 \rangle_E^*$$
 (3.39)

For boundary conditions of this type, (3.39) can be taken as the definition of losslessness, in the same way that  $\langle E_2, J_1 \rangle_R = \langle E_1, J_2 \rangle_R$  is the definition of reciprocity.

For a resonant frequency of a bounded region, (3.39) leads to an apparent contradiction. At a resonance, the electric field E associated with a source J is not uniquely defined by Eq. (3.17). For  $J_1 = J_2 = J$ ,  $E_1 = E$ , and  $E_2 = E + E_0$ , where  $E_0$  is a homogeneous solution to (3.17), Eq. (3.39) leads to the result that  $2\text{Re} < E, J >= - < E_0, J >^*$ . But (3.39) with  $E_1 = E_2 = E$  also requires that Re < E, J >= 0, so that  $< E_0, J >$ must vanish. The quantity  $< E_0, J >$ , however, is in general not zero, since the current J is arbitrary. The resolution of the contradiction lies in the observation that if J is not orthogonal to all resonant modes of V, power will continually be supplied to the field and the assumption of steady state fields upon which the results of this section depend becomes invalid.

Using the definition of L, Eq. (3.39) leads to

$$\int_{V} E_{1}^{*} \wedge (\tilde{\star}_{h} \mathbf{L} E_{2}) = \left[ \int_{V} E_{2}^{*} \wedge (\tilde{\star}_{h} \mathbf{L} E_{1}) \right]^{*}.$$
(3.40)

Thus, L is self-adjoint with respect to the inner product

$$\langle E_1, E_2 \rangle = \int_V E_1^* \wedge \check{\star}_h E_2 \tag{3.41}$$

as has been shown by Chew [52]. Replacing  $E_1$  with  $G(\mathbf{r}_1, \mathbf{r}_2)$  and  $E_2$  with  $G(\mathbf{r}_1, \mathbf{r}_3)$  in (3.40) then yields

$$G^*(\mathbf{r}_3, \mathbf{r}_2) = G(\mathbf{r}_2, \mathbf{r}_3) \tag{3.42}$$

so that the Green form for a lossless medium is hermitian if it satisfies boundary conditions such that the surface contribution of Eq. (3.37) vanishes.

#### **3.3** Green Form for the Anisotropic Helmholtz Equation

The derivative operator of Eq. (3.17) is not diagonal, making the solution for the Green form G difficult to obtain. Adding  $d\tilde{\star}_h d\star_h E$  to both sides of Eq. (3.17) yields

$$(\Delta_h + \omega^2 \star_h \star_e) E = -i\omega \star_h J + d\tilde{\star}_h d\star_h E \tag{3.43}$$

where  $\Delta_h$  is the wave operator in the metric due to the permeability of the medium and is defined using Eq. (3.14) to be  $-\star_h d\tilde{\star} d + d\tilde{\star}_h d\star_h$ . For a constant permeability tensor, the operator  $\Delta_h$  is diagonal and therefore simpler than the derivative operator of Eq. (3.17). Since the operator on the left-hand side reduces in free space essentially to the Helmholtz operator, I refer to (3.43) as the anisotropic Helmholtz equation. The corresponding Green  $1 \otimes 1$  form g satisfies

$$(\Delta_h + \omega^2 \star_h \star_e) g(\mathbf{r}_1, \mathbf{r}_2) = -\delta(\mathbf{r}_1 - \mathbf{r}_2) I$$
(3.44)

where operators act on the  $\mathbf{r}_1$  coordinate and I is the unit  $1 \otimes 1$  form. The Green form g can be found in closed form for certain types of media for which no exact solution for G is known. In free space,  $g = \mu_0^2 g_0 I$ , where  $g_0 = e^{ik_0 r}/(4\pi r)$  is the usual scalar Green function.

In Eq. (3.44) I have not included the  $\star_h$  operator on the right-hand side as was done in Eq. (3.18). Since  $\Delta_h$  is symmetric for constant  $\star_h$ , the additional  $\star_h$  operator does not simplify the derivations of this section as it did in Sec. 3.2. The formal transpose of gis defined to be the 2  $\otimes$  1 Green form  $\tilde{g}$  which satisfies

$$(\Delta_h + \omega^2 \tilde{\star}_e \tilde{\star}_h) \tilde{g}(\mathbf{r}_1, \mathbf{r}_2) = -\delta(\mathbf{r}_1 - \mathbf{r}_2) I$$
(3.45)

where I is the unit  $2 \otimes 1$  form. Note that the same derivative operator is employed in the definitions of g and  $\tilde{g}$ .

With the operators  $\mathbf{M} = (\Delta_h + \omega^2 \star_h \star_e)$  and  $\tilde{\mathbf{M}} = (\Delta_h + \omega^2 \tilde{\star}_e \tilde{\star}_h)$  appearing in the definitions of g and  $\tilde{g}$ , I seek to obtain a relationship of the form

$$C_1 \wedge \mathbf{M}E_2 - E_2 \wedge \mathbf{M}C_1 = dQ \tag{3.46}$$

where  $E_2$  is an arbitrary 1-form and  $C_1$  is an arbitrary 2-form. The conjunct Q of  $E_2$  and  $C_1$  defined by (3.46) can be shown to be

$$Q = \tilde{\star}_h C_1 \wedge \tilde{\star}_h dE_2 + \star_h d\tilde{\star}_h C_1 \wedge E_2 + C_1 \wedge \tilde{\star}_h d\star_h E_2 - \star_h dC_1 \wedge \star_h E_2.$$
(3.47)

Integrating (3.46) over a volume V and applying the generalized Stokes theorem yields

$$\int_{V} C_1 \wedge \mathbf{M} E_2 - \int_{V} E_2 \wedge \tilde{\mathbf{M}} C_1 = \int_{\partial V} Q.$$
(3.48)

Substituting  $\tilde{g}(\mathbf{r}_1, \mathbf{r}_2)$  for  $C_1(\mathbf{r})$  and using Eqs. (3.43) and (3.45), we have

$$E(\mathbf{r}_2) = i\omega \int_{V_1} \tilde{g}(\mathbf{r}_1, \mathbf{r}_2) \wedge \star_h J(\mathbf{r}_1) - \int_{V_1} \tilde{g}(\mathbf{r}_1, \mathbf{r}_2) \wedge d\tilde{\star}_h d\star_h E(\mathbf{r}_1) + \int_{\partial V_1} R \qquad (3.49)$$

where R is

$$R = \tilde{\star}_{h} \tilde{g}(\mathbf{r}_{1}, \mathbf{r}_{2}) \wedge \tilde{\star}_{h} dE(\mathbf{r}_{1}) + \star_{h} d\tilde{\star}_{h} \tilde{g}(\mathbf{r}_{1}, \mathbf{r}_{2}) \wedge E(\mathbf{r}_{1}) + \tilde{g}(\mathbf{r}_{1}, \mathbf{r}_{2}) \wedge \tilde{\star}_{h} d\star_{h} E(\mathbf{r}_{1}) - \star_{h} d\tilde{g}(\mathbf{r}_{1}, \mathbf{r}_{2}) \wedge \star_{h} E(\mathbf{r}_{1}).$$
(3.50)

Integrating the second term on the right–hand side of (3.49) twice by parts cancels two of the terms of R, leaving

$$E(\mathbf{r}_2) = i\omega \int_{V_1} \tilde{g}(\mathbf{r}_1, \mathbf{r}_2) \wedge \star_h J(\mathbf{r}_1) - \int_{V_1} \tilde{\star}_h d\star_h d\tilde{g}(\mathbf{r}_1, \mathbf{r}_2) \wedge E(\mathbf{r}_1) + \int_{\partial V_1} R_1 \qquad (3.51)$$

where the operator  $\tilde{\star}_h d\tilde{\star}_h d$  acts on the  $\mathbf{r}_1$  part of  $\tilde{g}$  and

$$R_1 = \check{\star}_h \tilde{g}(\mathbf{r}_1, \mathbf{r}_2) \wedge \check{\star}_h dE(\mathbf{r}_1) + \star_h d\check{\star}_h \tilde{g}(\mathbf{r}_1, \mathbf{r}_2) \wedge E(\mathbf{r}_1)$$
(3.52)

is the integrand of the surface contribution.

If  $\tilde{\star}_h \tilde{g}$  and E satisfy boundary conditions such as those described in Sec. 3.2.1, so that the surface integral term of (3.51) vanishes, then we obtain the relationship

$$E(\mathbf{r}_2) = i\omega \int_{V_1} \tilde{g}(\mathbf{r}_1, \mathbf{r}_2) \wedge \star_h J(\mathbf{r}_1) - \int_{V_1} \tilde{\star}_h d\star_h d\tilde{g}(\mathbf{r}_1, \mathbf{r}_2) \wedge E(\mathbf{r}_1).$$
(3.53)

This is a Fredholm integral equation of the second kind for the electric field in terms of the source J. As will be discussed in the following chapter, it may be possible to employ this equation as a basis for numerical techniques for treating scattering problems in complex media.

# **3.3.1** Integral Relationship Between G and $\tilde{g}$

Substituting  $\tilde{g}(\mathbf{r}_1, \mathbf{r}_2)$  for  $C_1$  and  $G(\mathbf{r}_1, \mathbf{r}_3)$  for  $E_2$  in Eq. (3.48) and following a procedure similar to the derivation of (3.51), I obtain the integral equation

$$G(\mathbf{r}_1, \mathbf{r}_2) = \tilde{\star}_h \tilde{g}(\mathbf{r}_2, \mathbf{r}_1) - \int_{V_3} \tilde{\star}_h d\star_h d\tilde{g}(\mathbf{r}_3, \mathbf{r}_1) \wedge G(\mathbf{r}_3, \mathbf{r}_2)$$
(3.54)

where the derivatives act on the  $\mathbf{r}_3$  coordinates of  $\tilde{g}$  and surface terms have been neglected. This expression generalizes to complex media the usual relationship between the scalar Green function and the Green form for isotropic, homogeneous media [3].

By repeated substitution on (3.54) I obtain a formal series solution for G,

$$G = \tilde{\star}_h \tilde{g} - \int \tilde{\star}_h d\star_h d\tilde{g} \wedge \tilde{\star}_h \tilde{g} + \int \int \tilde{\star}_h d\star_h d\tilde{g} \wedge \tilde{\star}_h d\star_h d\tilde{g} \wedge \tilde{\star}_h \tilde{g} - \cdots$$
(3.55)

where coordinate dependence is suppressed. For free space, up to factors of  $\mu_0$  the second term has components equal to the second partial derivatives of  $e^{ikr}/(8\pi ik)$ . Beyond the second term, increasingly high powers of r appear, so that for large r the series diverges. For a homogeneous medium, however, the wavevector representation of this series can be resummed, as will be shown in Sec. 3.5.

### 3.3.2 Symmetric Permeability Tensor

For a symmetric permeability tensor, one can simplify expressions (3.51) and (3.54) by absorbing a star operator  $\star_h$  into the definition of  $\tilde{g}$ . I therefore employ the modified definitions

$$(\Delta_h + \omega^2 \star_h \star_e)g = -\star_h \delta I \tag{3.56a}$$

$$(\Delta_h + \omega^2 \star_h \tilde{\star}_e)\tilde{g} = -\star_h \delta I \tag{3.56b}$$

where g and  $\tilde{g}$  are  $1 \otimes 1$  forms and I is the unit  $2 \otimes 1$  form. One can now obtain an identity of the form

$$E_1 \wedge \star_h \mathbf{M}' E_2 - E_2 \wedge \star_h \tilde{\mathbf{M}}' E_1 = dQ'$$
(3.57)

which replaces Eq. (3.46). By altering slightly the derivation given in the previous section, one can show that Eq. (3.51) simplifies to

$$E(\mathbf{r}_2) = i\omega \int_{V_1} \tilde{g}(\mathbf{r}_1, \mathbf{r}_2) \wedge J(\mathbf{r}_1) - \int_{V_1} \star_h d\star_h d\star_h \tilde{g}(\mathbf{r}_1, \mathbf{r}_2) \wedge E(\mathbf{r}_1).$$
(3.58)

The integral equation (3.54) becomes

$$\tilde{G}(\mathbf{r}_1, \mathbf{r}_2) = \tilde{g}(\mathbf{r}_1, \mathbf{r}_2) - \int_{V_3} \star_h d \star_h d \star_h \tilde{g}(\mathbf{r}_3, \mathbf{r}_2) \wedge \tilde{G}(\mathbf{r}_1, \mathbf{r}_3)$$
(3.59)

for a symmetric  $\star_h$  operator and the modified definitions (3.56).

#### 3.4 Electrically Inhomogeneous Media

The representation for the Green form (3.54), as well as the electric field integral equation (3.53), connect scalar scattering by an isotropic, magnetically homogeneous, electrically inhomogeneous medium with the electromagnetic scattering problem for the same medium. Solution of Eq. (3.44) for the Helmholtz Green form reduces to the determination of the usual scalar Green function  $g_s$  for the Helmholtz equation,

$$[\Delta + k^2(\mathbf{r}_1)]g_s(\mathbf{r}_1, \mathbf{r}_2) = -\delta(\mathbf{r}_1 - \mathbf{r}_2)$$
(3.60)

where  $k^2(\mathbf{r}) = \omega^2 \mu_0 \epsilon(\mathbf{r})$ . If the left-hand side of the definition (3.44) is multiplied by  $\mu_0^2$ , the Helmholtz Green form g is then equal to  $g_s I$ . The scalar Green function can be found analytically for certain types of inhomogeneous profiles, including the one-dimensional variation  $k^2(\mathbf{r}) = k_0^2(1 + az)$  [54] and the spherical profile  $k^2(\mathbf{r}) = k_0^2(1 + ar^2)$  [55], leading to exact, closed form solutions for the Helmholtz Green form. If the Helmholtz Green form is available in closed form, then the kernel of the integral equation (3.54) is known and the integral equation becomes an exact representation of the Green form for the electric field. The electric field integral equation (3.51) for electrically inhomogeneous media will be discussed further in the following chapter.

#### 3.5 Homogeneous Media

For a homogeneous medium, by spatial symmetry the components of g are shift-invariant functions  $g_{ij}(\mathbf{r}_1 - \mathbf{r}_2)$ . The integral in (3.54) becomes a convolution, so that the Fourier transform of the integral is the product of the transforms of  $\tilde{\star}_h d\star_h d\tilde{g}$  and G. The following transform relations can be used to obtain the wavevector representation of  $\tilde{\star}_h d\star_h d\tilde{g}$ . If  $a(\mathbf{r}_1, \mathbf{r}_2)$  is a  $2 \otimes 1$  form,  $b(\mathbf{r}_1, \mathbf{r}_2)$  is a  $1 \otimes 1$  form, and  $c(\mathbf{r}_1, \mathbf{r}_2)$  is a 0-form in the  $\mathbf{r}_1$  coordinates, then

$$da(\mathbf{r}_1, \mathbf{r}_2) \longleftrightarrow i\mathbf{k}^T a(\mathbf{k})$$
$$\tilde{\star}_h b(\mathbf{r}_1, \mathbf{r}_2) \longleftrightarrow \mu^T b(\mathbf{k})$$
$$dc(\mathbf{r}_1, \mathbf{r}_2) \longleftrightarrow i\mathbf{k} c(\mathbf{k})$$

where k is the wavevector and derivatives act on the  $r_1$  coordinates. The coefficients of the forms are functions of  $r_1 - r_2$ , and the spatial Fourier transform is taken with respect

to this quantity. Throughout this section, the Fourier transforms of double forms will be represented for convenience as matrices, and the same symbol used for both the physical space representation of a double form and the matrix of the transforms of its components. Matrix components are ordered according to Eq. (3.19).

The Fourier transform of  $\tilde{\star}_h d\star_h d\tilde{g}$  becomes  $-(1/\det \mu)\mu^T \mathbf{k} \mathbf{k}^T \tilde{g}$ . Since the exterior product of this term with *G* acts on the first argument part rather than the second, the matrix must be transposed in order to obtain the transform of the product. The spatial Fourier transform of Eq. (3.54) is therefore

$$G = \tilde{g}^T \mu^{-1} + \frac{1}{\det \mu} \tilde{g}^T \mathbf{k} \mathbf{k}^T \mu G.$$
(3.61)

Solving for G, we obtain

$$G = \left[\mu \tilde{g}^{T-1} - \frac{1}{\det \mu} \mu \mathbf{k} \mathbf{k}^T \mu\right]^{-1}.$$
(3.62)

Similarly, the Fourier transform of Eq. (3.45) shows that

$$\tilde{g} = \left[\frac{1}{\det\mu} (\mathbf{k}^T \mu \mathbf{k}) \mathbf{I} - \omega^2 \epsilon^T \mu^{T-1}\right]^{-1}$$
(3.63)

where I is the identity matrix. Eq. (3.18) leads to an alternate expression for G,

$$G = \left[-\Gamma \mu^{T-1} \Gamma - \omega^2 \epsilon\right]^{-1}$$
(3.64)

where

$$\Gamma = \begin{bmatrix} 0 & -k_z & k_y \\ k_z & 0 & -k_x \\ -k_y & k_x & 0 \end{bmatrix}.$$
 (3.65)

Substituting (3.63) into (3.62) gives

$$G = \left[ -\frac{1}{\det \mu} \mu \mathbf{k} \mathbf{k}^T \mu + \frac{1}{\det \mu} (\mathbf{k}^T \mu \mathbf{k}) \mu - \omega^2 \epsilon \right]^{-1}$$
(3.66)

which is equivalent to the result obtained in Ref. [9]. The poles of G in the wavevector representation represent plane wave solutions to (3.17), so that

$$\det\left[-\frac{1}{\det\mu}\mathbf{k}\mathbf{k}^{T}\mu + \frac{1}{\det\mu}(\mathbf{k}^{T}\mu\mathbf{k})\mathbf{I} - \omega^{2}\mu^{-1}\epsilon\right] = 0$$
(3.67)

is the Fresnel equation [9, 56] for an arbitrary homogeneous medium.

Finally, if the wave operator  $\Delta_h$  is applied to a 1-form, the Fourier transform of the definition (3.14) becomes

$$-\frac{1}{\det\mu}(\mathbf{k}^{T}\mu\mathbf{k})\mathbf{I} = \mu^{-1}\Gamma\mu^{T-1}\Gamma - \frac{1}{\det\mu}\mathbf{k}\mathbf{k}^{T}\mu.$$
(3.68)

This expression shows explicitly that the operator  $\Delta_h$  is diagonal for an arbitrary (constant) permeability tensor.

The wavevector representation of the series solution (3.55) for G can be resummed, leading to another representation for the Green form for the electric field. By resubstitution, Eq. (3.61) can be transformed into

$$G = \tilde{g}^T \mu^{-1} + \frac{1}{\det \mu} \tilde{g}^T \mathbf{k} \mathbf{k}^T \mu \tilde{g}^T \mu^{-1} + \frac{1}{\det \mu} \tilde{g}^T \mathbf{k} \mathbf{k}^T \mu \frac{1}{\det \mu} \tilde{g}^T \mathbf{k} \mathbf{k}^T \mu \tilde{g}^T \mu^{-1} + \cdots \quad (3.69)$$

This can be rewritten as

$$G = \tilde{g}^T \mu^{-1} + \frac{1}{\det \mu} \tilde{g}^T \mathbf{k} \left[ 1 + \frac{1}{\det \mu} \mathbf{k}^T \mu \tilde{g}^T \mathbf{k} + \left( \frac{1}{\det \mu} \mathbf{k}^T \mu \tilde{g}^T \mathbf{k} \right)^2 + \cdots \right] \mathbf{k}^T \mu \tilde{g}^T \mu^{-1}$$
(3.70)

where the series is now scalar and geometric. Summing the series yields

$$G = \tilde{g}^T \mu^{-1} + \frac{1}{\det \mu} \left[ \frac{\tilde{g}^T}{1 - \frac{1}{\det \mu} \mathbf{k}^T \mu \tilde{g}^T \mathbf{k}} \right] \mathbf{k} \mathbf{k}^T \mu \tilde{g}^T \mu^{-1}.$$
 (3.71)

For free space, the quantity inside square brackets is equal to  $\mu_0^2 I/k_0^2$ , so that this expression in physical space reduces up to factors of  $\mu_0$  to the usual expression for the Green form for the electric field. The series in Eq. (3.70) is singular for values of k that represent allowed plane waves, so that

$$\frac{1}{\det \mu} \mathbf{k}^T \mu \tilde{g}^T \mathbf{k} = 1$$
(3.72)

is equivalent to the Fresnel equation (3.67).

## 3.5.1 Exact Solution for the Helmholtz Green Form

If  $\mu_{ij}$  is diagonalizable by a rotation, then the inverse transform of  $\tilde{g}(\mathbf{k})$  can be obtained in closed form. In this case, the kernel of the electric field integral equation (3.53) and the Green form integral relationship (3.54) is known exactly. From Eq. (3.63),

$$\tilde{g}(\mathbf{r}_1, \mathbf{r}_2) = \frac{1}{(2\pi)^3} \int d\mathbf{k} e^{i\mathbf{k}\cdot\mathbf{r}} \left[ \frac{1}{\det\mu} (\mathbf{k}^T \mu \mathbf{k}) \mathbf{I} - \omega^2 \epsilon^T \mu^{T-1} \right]^{-1}$$
(3.73)

where  $\mathbf{r} = \mathbf{r}_1 - \mathbf{r}_2$ . By rotating the coordinate system so that  $\mu_{ij}$  is diagonal and performing a further change of variables such that  $k'^2 = \mathbf{k}^T \mu \mathbf{k} / (\det \mu)$ , this simplifies to

$$\tilde{g}(\mathbf{r}_1, \mathbf{r}_2) = \frac{\det \mu}{(2\pi)^3} \int d\mathbf{k}' e^{i\mathbf{k}' \cdot \tilde{\mathbf{r}}} \left[ k'^2 \mathbf{I} - \omega^2 \epsilon^T \mu^{T-1} \right]^{-1}$$
(3.74)

where  $\tilde{\mathbf{r}} = \sqrt{\det \mu} (\hat{\mathbf{x}}x/\sqrt{\mu_1} + \hat{\mathbf{y}}y/\sqrt{\mu_2} + \hat{\mathbf{z}}z/\sqrt{\mu_3})$ . Here, x, y, and z are the components of  $\mathbf{r}$  and  $\mu_1, \mu_2$ , and  $\mu_3$  are the eigenvalues of  $\mu_{ij}$ . I assume for convenience that the eigenvalues of  $\mu_{ij}$  are positive and real. Rotating  $\mathbf{k}'$  so that  $k'_z$  is in the  $\tilde{\mathbf{r}}$  direction, we obtain

$$\tilde{g}(\mathbf{r}_1, \mathbf{r}_2) = \frac{\det \mu}{(2\pi)^3} \int k^{\prime 2} \sin \theta \, dk^{\prime} \, d\theta \, d\phi \, e^{ik^{\prime}\tilde{r}\cos\theta} \left[k^{\prime 2}\mathbf{I} - \omega^2\epsilon^T\mu^{T-1}\right]^{-1}$$
(3.75)

where  $\theta$  and  $\phi$  are the angles associated with k'. By integrating the angles,

$$\tilde{g}(\mathbf{r}_1, \mathbf{r}_2) = \frac{\det \mu}{4i\pi^2 \tilde{r}} \int k' dk' \left( e^{ik'\tilde{r}} - e^{-ik'\tilde{r}} \right) \left[ k'^2 \mathbf{I} - \omega^2 \epsilon^T \mu^{T-1} \right]^{-1}.$$
(3.76)

The remaining k' integration can be performed if  $\epsilon^T \mu^{T-1}$  has a square root.

The matrix  $\epsilon^T \mu^{T^{-1}}$  is not in general diagonalizable (see Ref. [9]), but it has a Jordan normal form  $SJS^{-1}$ . Consider one of the Jordan blocks of J, corresponding to the eigenvalue  $ae^{ib}$  where a and b are positive and real. For this block, I construct the square root

$$\begin{bmatrix} ae^{ib} & 1 \\ & \ddots & \ddots \\ & & \ddots & 1 \\ & & & ae^{ib} \end{bmatrix}^{1/2} = \pm \begin{bmatrix} \sqrt{a}e^{ib/2} & 1/(2\sqrt{a}e^{ib/2}) \\ & \ddots & \ddots \\ & & \ddots & 1/(2\sqrt{a}e^{ib/2}) \\ & & \sqrt{a}e^{ib/2} \end{bmatrix}$$
(3.77)

where the sign is chosen so that  $\operatorname{Re}\{\pm\sqrt{a}e^{ib/2}\}$  is positive. As with the case that  $\epsilon^T\mu^{T^{-1}}$  is diagonalizable, the other root can be discarded by causality. (If the eigenvalues of  $\mu_{ij}$  are not positive and real, then determination of the outgoing solution is more difficult since  $\tilde{\mathbf{r}}$ also becomes complex and nonunique.) The right–hand side of (3.77) exists since  $\epsilon_{ij}$  and  $\mu_{ij}$  are by assumption invertible, so that  $\epsilon^T\mu^{T^{-1}}$  has no zero eigenvalues.

By proceeding in this manner for each block of J, I construct  $J^{1/2}$ , so that  $K = \omega S J^{1/2} S^{-1}$  is a square root of  $\omega^2 \epsilon^T \mu^{T^{-1}}$ . Equation (3.76) then becomes

$$\tilde{g}(\mathbf{r}_{1},\mathbf{r}_{2}) = \frac{\det \mu}{8i\pi^{2}\tilde{r}} \int_{0}^{\infty} dk' \left(e^{ik'\tilde{r}} - e^{-ik'\tilde{r}}\right) \left[\left(k'\mathbf{I} - K\right)^{-1} + \left(k'\mathbf{I} + K\right)^{-1}\right].$$
(3.78)

This can be rewritten as

$$\tilde{g}(\mathbf{r}_{1},\mathbf{r}_{2}) = \frac{\det \mu}{8i\pi^{2}\tilde{r}} \int_{-\infty}^{\infty} dk' \, e^{ik'\tilde{r}} \left[ \left(k'\mathbf{I} - K\right)^{-1} - \left(k'\mathbf{I} + K\right)^{-1} \right].$$
(3.79)

Since the eigenvalues of K have positive real part, the second term can be discarded by causality, and the final result is

$$\tilde{g}(\mathbf{r}_1, \mathbf{r}_2) = (\det \mu) \frac{e^{iK\tilde{r}}}{4\pi\tilde{r}}$$
(3.80)

for the transposed Helmholtz Green form with a radiation boundary condition. This can be seen to be a direct generalization of the free space scalar Green function  $e^{ikr}/(4\pi r)$ .

The matrix exponential in Eq. (3.80) can be computed in closed form from the Jordan normal form of K, so that  $\tilde{g}$  can be obtained explicitly. There are three possible cases for the normal form J, depending on the number of unique eigenvalues. If there is only one unique eigenvalue  $ae^{ib}$ , then  $J^{1/2}$  can be written as

$$J^{1/2} = \begin{bmatrix} m & 1/(2m) \\ m & 1/(2m) \\ & m \end{bmatrix}$$
(3.81)

where  $m = \pm \sqrt{a}e^{ia/2}$  with the sign chosen as described above. The transposed Helmholtz Green form then has components

$$\tilde{g} = (\det \mu) \frac{e^{im\tilde{r}}}{4\pi\tilde{r}} S \begin{bmatrix} 1 & 1/(2m) & 1/(8m^2) \\ 1 & 1/(2m) \\ & 1 \end{bmatrix} S^{-1}.$$
(3.82)

If  $\epsilon^T \mu^{T^{-1}}$  has two unique eigenvalues,  $a_1 e^{ib_1}$  with multiplicity two and  $a_2 e^{ib_2}$  with multiplicity one, then

$$J^{1/2} = \begin{bmatrix} m_1 & 1/(2m_1) & \\ & m_1 & \\ & & m_2 \end{bmatrix}$$
(3.83)

where  $m_1 = \pm \sqrt{a_1} e^{ia_1/2}$  and  $m_2 = \pm \sqrt{a_2} e^{ia_2/2}$  with the signs chosen individually so that  $m_1$  and  $m_2$  both have positive real part. In this case,  $\tilde{g}$  becomes

$$\tilde{g} = (\det \mu) \frac{1}{4\pi \tilde{r}} S \begin{bmatrix} e^{im_1 \tilde{r}} & e^{im_1 \tilde{r}} / (2m_1) \\ e^{im_1 \tilde{r}} \\ e^{im_1 \tilde{r}} \end{bmatrix} S^{-1}.$$
(3.84)

Finally, if  $\epsilon^T \mu^{T^{-1}}$  has three unique eigenvalues, there is a coordinate system for which  $\epsilon^T \mu^{T^{-1}}$  is diagonal. In that coordinate system, we have that

$$\tilde{g} = (\det \mu) \frac{1}{4\pi \tilde{r}} \begin{bmatrix} e^{im_1 \tilde{r}} & \\ & e^{im_2 \tilde{r}} \\ & & e^{im_3 \tilde{r}} \end{bmatrix}$$
(3.85)

where  $m_1, m_2$ , and  $m_3$  are the square roots of the eigenvalues with positive real part.

As we have just seen, if  $\epsilon^T \mu^{T^{-1}}$  is diagonal, then  $\tilde{g}$  is also diagonal. For symmetric or hermitian  $\epsilon_{ij}$  and  $\mu_{ij}$ , this is equivalent to the simultaneous diagonalizability of  $\epsilon_{ij}$  and  $\mu_{ij}$ . A commonly encountered type of medium for which  $\epsilon^T \mu^{T^{-1}}$  is diagonal is a biaxial material, which is a homogeneous, magnetically isotropic medium such that the permittivity tensor has unique eigenvalues. For convenience, I scale G by a factor of  $\mu_0$  and g by a factor of  $\mu_0^2$ . If the coordinates system is transformed so that the permittivity tensor is diagonal with eigenvalues  $\epsilon_i$ , then  $g(\mathbf{k})$  has the diagonal components

$$g_{ii}(\mathbf{k}) = \frac{1}{k^2 - k_{0i}^2} \tag{3.86}$$

where  $k_{0i} = \omega \sqrt{\epsilon_i \mu_0}$  and other elements vanish. In physical space,

$$g(\mathbf{r}_1, \mathbf{r}_2) = \frac{e^{ik_{01}r}}{4\pi r} \, dx_1 \otimes \, dx_2 + \frac{e^{ik_{02}r}}{4\pi r} \, dy_1 \otimes \, dy_2 + \frac{e^{ik_{03}r}}{4\pi r} \, dz_1 \otimes \, dz_2 \tag{3.87}$$

where  $r = |\mathbf{r}_1 - \mathbf{r}_2|$ . The representation (3.71) of the Green form for the electric field becomes

$$G = g + \left(\frac{g}{1 - \mathbf{k}^T g \mathbf{k}}\right) \mathbf{k} \mathbf{k}^T g$$
(3.88)

and the Fresnel equation can be written as  $\mathbf{k}^T g \mathbf{k} = 1$  for a biaxial medium.

## 3.6 Summary

In order to conveniently represent macroscopic electromagnetic properties of a medium, I have defined anisotropic Hodge star operators in which the permittivity and permeability tensors of the medium are embedded. The use of these operators along with other tools of the calculus of differential forms makes expressions concise and simplifies manipulations. Because the physical meaning of a quantity is contained in the degree of the differential form, rather than in the type of derivative operator acting on it, a few general properties and theorems suffice for the derivations given in this chapter. This strength is also a weakness, since final expressions with many exterior derivatives and star operators are impossible to interpret physically without knowing the degrees of the differential forms involved and keeping track of changes in degree as operators are applied. For derivations such as those performed in this chapter, however, differential forms are ideal.

The use of electric and magnetic star operators to express the constitutive relations leads to a natural generalization of the free space scalar Green function, which I have called the Helmholtz Green form. The main result of this chapter is an integral equation connecting the Helmholtz Green form to the Green form for the electric field. This integral equation extends to complex media the well–known construction of the free space tensor Green function from the scalar Green function. The Helmholtz Green form with a radiation boundary condition can be obtained exactly in physical space for a homogeneous medium with diagonalizable permittivity tensor, and is essentially equivalent to the scalar Green function for an isotropic, magnetically homogeneous medium. Also obtained is an integral equation for the electric field in terms of the Helmholtz Green form, which will be examined in greater detail in the next chapter.

## Chapter 4

# **ELECTRIC FIELD INTEGRAL EQUATION**

The integral equation obtained in the previous chapter for the electric field in terms of sources, field boundary values, and the Helmholtz Green form,

$$E = i\omega \int_{V} \tilde{\star}_{h} \tilde{g} \wedge J - \int_{V} \tilde{\star}_{h} d\star_{h} d\tilde{g} \wedge E + \int_{\partial V} \left( \tilde{\star}_{h} \tilde{g} \wedge \tilde{\star}_{h} dE + \star_{h} d\tilde{\star}_{h} \tilde{g} \wedge E \right)$$
(4.1)

is valid for an arbitrarily anisotropic, inhomogeneous medium. If boundary conditions are such that the surface integral term can be neglected, this is a Fredholm integral equation of the second kind. The integral equation may be useful as a basis for numerical methods for computing scattered fields or could provide theoretical insights for various problems if exact or asymptotic solution methods can be found. In this chapter, I examine this integral equation, its possible applications, and relationship to other representations for the electric field in free space and complex media.

Section 4.1 outlines possible applications of this integral equation and presents arguments as to problems for which it may be superior to the usual integral equation methods. In Sec. 4.2, the standard free space expression for the electric field in terms of a scalar Green function is manipulated into a form such that the integral equation (4.1) can be seen to be a direct generalization. A principal value interpretation for the volume integration of (4.1) is vital to its numerical evaluation. This is considered further in Sec. 4.3.

For clarity, coordinate dependence will be suppressed in nearly all expressions. Integrals are over the  $\mathbf{r}_1$  coordinates unless otherwise noted. Any quantity under an integral which is not a double form will depend on the  $\mathbf{r}_1$  coordinates. Operators under integrals generally act on the  $\mathbf{r}_1$  coordinates. Those few operators appearing outside of integrals generally operate on quantities which are not double forms, and so there is no ambiguity.

#### 4.1 Applications

For a given geometry, medium, and boundary condition, the integral equation (4.1) can be applied directly only if the Helmholtz Green form g is known. As noted in the

previous chapter, the Helmholtz Green form can be found in closed form for an unbounded, homogeneous, anisotropic medium. For an electrically inhomogeneous, isotropic medium, the defining equation for the Helmholtz Green form essentially reduces to the usual scalar Helmholtz equation. These two classes of media include many of the types of problems which are of interest in applications, and will be discussed further in this section.

There are media which do not fall into these two groups, such as isotropic, magnetically inhomogeneous media and materials which are both anisotropic and homogeneous. The former case can be treated by duality, thereby reducing the problem to that of an electrically inhomogeneous medium. Media with spatially varying properties as well as anisotropy might be subdivided by types of symmetry, and methods of solving for the Helmholtz Green form based on the symmetries of the medium then sought.

#### 4.1.1 Homogeneous Media

For an unbounded, homogeneous, anisotropic medium with a radiation boundary condition, the exact solution (3.80) for the Helmholtz Green form leads to a closed form representation of the kernel of (4.1). If the radiation boundary condition is of the form of (3.30), the surface contribution of vanishes as well, so that (4.1) can in principle be employed to solve for the electric field due to a given source. For a bounded region, a solution for the Helmholtz Green form satisfying an appropriate boundary condition would be required in order to apply the integral equation.

Previous integral equation methods for scattering by electrically anisotropic media rely on the use of an equivalent source which depends on the electric field. For such media, the electric field satisfies

$$(-\star d \star d + k_0^2)E = -i\omega\mu_0 \star J - \omega^2\mu_0 \star \star_e'E \tag{4.2}$$

where  $k_0^2 = \omega^2 \mu_0 \epsilon_0$  and  $\star_{e'}$  is defined similarly to  $\star_e$  but with the permittivity taken to be  $\epsilon_{ij}(\mathbf{r}) - \epsilon_0 \delta_{ij}$ . For a homogeneous, isotropic medium, the results of Sec. 3.2 for the electric field in terms of the Green form G simplify to [3]

$$E = i\omega\mu_0 \int_V G_0 \wedge J + \int_{\partial V} \left( G_0 \wedge \star dE + \star dG_0 \wedge E \right)$$
(4.3)

where  $G_0$  is the double  $1 \otimes 1$  form given by

$$G_0 = \left(1 + \frac{1}{k_0^2} d\star d\star\right) g_0 I. \tag{4.4}$$

The equivalent source of Eq. (4.2) can be written  $J' = J - i\omega \star_e' E$ , so that the electric field satisfies the integral equation

$$E = i\omega\mu_0 \int_V G_0 \wedge J - i\omega \int_V G_0 \wedge \star_e' E \tag{4.5}$$

assuming boundary conditions such that the surface term vanishes. This integral equation has long been used as a numerical method for computation of fields in electrically anisotropic and inhomogeneous media [57, 58]. The first term on the right (the "incident field") becomes equal to the exact electric field as  $\epsilon_{ij}(\mathbf{r})$  approaches  $\epsilon_0 \delta_{ij}$ . Thus, for small anisotropy, (4.5) can be solved efficiently by using the first few terms of the Neumann series solution. For large anisotropy, the integral equation is more difficult to deal with.

For an electrically anisotropic medium, Eq. (4.1) can be simplified to

$$E = i\omega\mu_0 \int_V g \wedge J - \int_V \star d\star d\star g \wedge E.$$
(4.6)

For a biaxial medium with radiation boundary conditions, the incident field term gives the exact electric field if J represents a plane current oriented perpendicular to any of the three principal axes of the permittivity tensor. The corresponding term of (4.5) can produce the exact electric field for a plane wave propagating in at most one direction. In this sense, the incident field term of (4.6) is a more accurate approximation to the true electric field. In effect, the incident field term of (4.5) approximates the wave surface of the medium by a single sphere of radius  $k_0$ , while the incident field term of (4.6) implies a wave surface consisting of three spheres of radii  $k_{01}$ ,  $k_{02}$ , and  $k_{03}$ . The true wave surface consists of two sheets with portions lying near each of these three spheres [7]. This may lead to an advantage when solving (4.6) numerically for the case of large anisotropy.

## 4.1.2 Inhomogeneous Media

As discussed in Sec. 3.4, finding the Helmholtz Green form for an electrically inhomogeneous, isotropic medium reduces to the determination of the Green function  $g_s$  for

the scalar Helmholtz equation. Thus, the integral equation (4.1) connects the scalar Green function for media of this type with the scattering problem for the electric field E, so that exact or asymptotic results for vector scattering can be obtained from a knowledge of the scalar scattering. Exact solutions for the scalar Green function are known for media with certain types of permittivity profiles; examples are given in Sec. 3.4. These solutions in turn yield exact representations for the kernel of the electric field integral equation (4.1). For a general permittivity profile, a numerical solution to the scalar problem could be employed to approximate the kernel of (4.1).

For a medium with slowly varying permittivity, the integral equation (4.1) may be more efficient as a solution method than the usual integral equation (4.5). In order to demonstrate this, we begin with the wave equation for the electric field in the form

$$[\Delta + k^2(\mathbf{r})]E = -i\omega\mu_0 \star J + d\star d\star E.$$
(4.7)

Gauss's law requires that

$$\rho = d\epsilon(\mathbf{r}) \star E$$
$$= (d\epsilon) \wedge \star E + \epsilon d \star E$$

If the spatial variation of the permittivity is much slower than the change in phase of the electric field, then the first term on the right-hand side of this expression can be neglected and we have that  $d\star E \simeq \rho/\epsilon$ . Equation (4.7) then shows that each component of the electric field in the slowly varying approximation satisfies the Helmholtz equation

$$[\Delta + k^2(\mathbf{r})]E = -i\omega\mu_0 \star J' \tag{4.8}$$

where  $J' = J - \star d \star [\rho/(i\omega\mu_0\epsilon)]$ . It would therefore be expected that the electric field can be approximated by an expression of the form  $E = i\omega\mu_0 \int g \wedge J'$ . This is indeed the case, as will be seen below.

Using the results of Sec. 3.3 specialized to an isotropic, magnetically homogeneous medium, we have from (3.49) that

$$E = i\omega\mu_0 \int_V g \wedge J - \int_V g \wedge \star d \star d \star E + \int_{\partial V} [g \wedge \star dE + \star dg \wedge E + (\star d \star g) \star E - \star g(\star d \star E)]$$
(4.9)

where g is equal to  $g_s I$  with I the unit  $1 \otimes 1$  form. Integrating the second term on the right by parts yields

$$E = i\omega\mu_0 \int_V g \wedge J + \int_V (\star d\star g)d\star E + \int_{\partial V} [g \wedge \star dE + \star dg \wedge E - (\star d\star g)\star E]$$
(4.10)

Using Gauss's law and integration by parts, this can be rewritten as

$$E = i\omega\mu_0 \int_V g \wedge J' + \int_V (\star d\star g) \left(\frac{d\epsilon}{\epsilon} \wedge \star E\right) + \int_{\partial V} [g \wedge \star dE + \star dg \wedge E - (\star d\star g) \star E + (\star g) \star \rho/\epsilon].$$
(4.11)

The first two terms of the surface contribution are zero for magnetically conducting, electrically conducting, or radiation boundary conditions. For magnetically conducting boundary conditions, the third surface term vanishes approximately for a slowly varying medium, since  $d \star g = d \star E \simeq \rho/\epsilon$  and  $\rho = 0$  on the boundary, and the fourth term vanishes exactly. The third and fourth terms also do not contribute for radiation boundary conditions as well, since  $\rho = 0$  at infinity. If in addition to the magnetically conducting or radiation boundary condition the medium is homogeneous near the boundary or at infinity, all of the surface terms vanish identically. If the boundary terms vanish, we obtain the integral equation

$$E = i\omega\mu_0 \int_V g \wedge J' + \int_V (\star d\star g) \left(\frac{d\epsilon}{\epsilon} \wedge \star E\right)$$
(4.12)

for the electric field. Note that J' can be written as  $J + \star d \star (1/k^2(\mathbf{r})) dJ$ , so that if the medium is homogeneous, then Eq. (4.12) is equivalent up to boundary contributions to the usual result for the electric field in terms of the free space scalar Green function.

If the wavelength is much smaller than the scale of spatial variation of the medium, then the electric field is given approximately by the first term on the right of (4.12). The volume integral term of (4.12) involving the unknown field E contributes only a small correction to the total electric field. Determination of the unknown field E from the source J thus requires inversion of a well–conditioned integral operator. The series solution for the integral equation will be rapidly convergent as well. By contrast, for a medium which is slowly varying but not weakly inhomogeneous, the incident field term of the usual integral equation method,  $i\omega\mu_0 \int_V G_0 \wedge J$ , is a poor approximation to the true field. The free space Green form  $G_0$  contains no information about the variation of the permittivity of the medium. Because of this, the integral equation (4.12) should be superior to

the usual method for strongly varying dielectric profiles admitting an exact solution for the scalar Green function. Even for inhomogeneous media without an exact Green function, a two step approach in which the scalar Green function is first obtained numerically and Eq. (4.12) then solved to find the electric field could have advantage over direct use of the usual volume integral equation, since efficient computational methods for determination of the Green function for the scalar Helmholtz equation are available [19].

# 4.2 Correspondence with Free Space Results

In this section, I show the relationship between the integral equation (4.1) and the well–known expression for the electric field in terms of the scalar Green function for an isotropic, homogeneous medium such as free space. In free space, the electric potential  $\phi$  in the Lorentz gauge  $\star d \star A = i\omega \epsilon_0 \mu_0 \phi$  satisfies

$$(\Delta + k_0^2)\phi = -\star\rho/\epsilon_0 \tag{4.13}$$

and the magnetic potential 1-form A satisfies

$$(\Delta + k_0^2)A = -\mu_0 \star J \tag{4.14}$$

where A is defined to be a 1-form such that B = dA,  $\phi$  is a 0-form which satisfies  $E = i\omega A - d\phi$ , and the constant  $k_0^2$  is equal to  $\omega^2 \epsilon_0 \mu_0$ .

These two expressions show that the electric potential and each component of A in rectangular coordinates obey a scalar Helmholtz equation of the form

$$(\Delta + k_0^2)u(\mathbf{r}) = f(\mathbf{r}). \tag{4.15}$$

In order to solve this differential equation, one defines a scalar Green function  $g_0$  such that

$$(\Delta + k_0^2)g_0(\mathbf{r}_1, \mathbf{r}_2) = -\delta(\mathbf{r}_1 - \mathbf{r}_2).$$

$$(4.16)$$

I will denote by  $M_0$  the differential operator on the left of this equation. From the definition of  $\Delta$ , if u is a 0-form, then  $\Delta u = \star d \star du$ . Using this result along with the product rule for the exterior derivative, we have for arbitrary 0-forms  $u_1$  and  $u_2$ ,

$$\star u_1 \mathbf{M}_0 u_2 - \star u_2 \mathbf{M}_0 u_1 = u_1 d \star d u_2 - u_2 \star d \star d u_1$$
  
=  $d(u_1 \star d u_2 - u_2 \star d u_1).$  (4.17)

This expression leads to a scalar form of Green's theorem for the operator  $M_0$ . By integrating (4.17) over a volume V and applying Stokes theorem, we obtain

$$\int_{V} \left( \star u_1 \mathbf{M}_0 u_2 - \star u_2 \mathbf{M}_0 u_1 \right) = \int_{\partial V} \left( u_1 \star du_2 - u_2 \star du_1 \right). \tag{4.18}$$

Setting  $u_1 = u(\mathbf{r}_1)$  and  $u_2 = g_0(\mathbf{r}_1, \mathbf{r}_2)$  in this expression yields the result

$$u(\mathbf{r}_{2}) = -\int_{V_{1}} \star g_{0}(\mathbf{r}_{1}, \mathbf{r}_{2}) f(\mathbf{r}_{1}) + \int_{\partial V_{1}} \left[ u(\mathbf{r}_{1}) \star dg_{0}(\mathbf{r}_{1}, \mathbf{r}_{2}) - g_{0}(\mathbf{r}_{1}, \mathbf{r}_{2}) \star du(\mathbf{r}_{1}) \right].$$
(4.19)

Using this result, we can write that

$$\phi = \int_{V} g_0 \rho / \epsilon_0 + \int_{\partial V} \left( g_0 \star d\phi - \phi \star dg_0 \right)$$
(4.20)

$$A_i = \int_V \star g_0 \mu_0 J_i + \int_{\partial V} \left( g_0 \star dA_i - A_i \star dg_0 \right)$$
(4.21)

where the coordinate dependence is suppressed and the components  $A_i$  are treated as 0forms. The relationship  $E = i\omega A - d\phi$ , allows the electric field to be written in terms of (4.20) and (4.21), so that

$$E = i\omega\mu_0 \int_V \star g_0 J_i - d \int_V g_0 \rho / \epsilon_0 + i\omega \int_{\partial V} (g_0 \star dA_i - A_i \star dg_0) - d \int_{\partial V} (g_0 \star d\phi - \phi \star dg_0)$$
(4.22)

for the electric field in terms of the source J and the boundary values of  $\phi$  and A.

The expression (4.22) appears to be different from the isotropic reduction of (4.1), but I will demonstrate the equivalence of the two formulations. This can be done most conveniently by employing the intermediate step of expressing  $d\phi$  and A in terms of the Helmholtz Green form g, rather than the scalar Green function  $g_0$ . For free space, the definition (3.44) of the previous chapter can be simplified to

$$(\Delta + k_0^2)g(\mathbf{r}_1, \mathbf{r}_2) = -\delta(\mathbf{r}_1 - \mathbf{r}_2)I$$
(4.23)

where the derivative operator acts on the  $\mathbf{r}_1$  coordinates and I is the unit double  $1 \otimes 1$  forms. With this definition, g is equal to  $g_0I$ .

The relationship  $d_2g_0(\mathbf{r}_1, \mathbf{r}_2) = -\star_1 d_1 \star_1 g(\mathbf{r}_1, \mathbf{r}_2)$ , where subscripts on the operators indicate the coordinates on which the operators act, follows from the translational invariance of the free space Green function. Using this result together with (4.20) shows that  $d\phi$  can be written as

$$d\phi = -\int_{V} (\star d \star g) \rho / \epsilon_0 - \int_{\partial V} \left[ (\star d \star g) \star d\phi - \phi \star d \star d \star g \right].$$
(4.24)

The magnetic potential A satisfies Eq. (4.14), which is similar in form to Eq. (3.43) of the previous chapter. The derivation of Sec. 3.3 therefore can be employed to show that

$$A = \mu_0 \int_V g \wedge J + \int_{\partial V} \left[ g \wedge \star dA + \star dg \wedge A + (\star g) \star d \star A - (\star A) \star d \star g \right].$$
(4.25)

Combining this expression with (4.24) for  $d\phi$  shows that the electric field can be expressed as

$$E = i\omega\mu_0 \int_V g \wedge J + \int_V (\star d\star g)\rho/\epsilon_0 + \int_{\partial V} S$$
(4.26)

where the  $2 \otimes 1$  form S is given by

$$S = i\omega g \wedge \star dA + i\omega \star dg \wedge A + i\omega (\star g) \star d \star A - i\omega (\star A) \star d \star g + (\star d \star g) \star d\phi - \phi \star d \star d \star g$$
(4.27)

and represents the surface contribution. It remains to demonstrate that Eq. (4.22) is equivalent to this result and in turn that (4.26) is equivalent to the free space special case of the integral equation (4.1).

The volume integral terms of Eqs. (4.22) and (4.26) are easily seen to be equal. The surface integral terms involving  $\phi$  are also clearly identical. All that remains to compare between the two expressions are the surface integral terms involving A. Leaving out a factor of  $i\omega$ , the  $dx_2$  component of the surface integrand due to A of Eq. (4.22) is  $(g_0A_{1x} - A_1g_{0x}) dy_1 dz_1 + (g_0A_{1y} - A_1g_{0y}) dz_1 dx_1 + (g_0A_{1z} - A_1g_{0z}) dx_1 dy_1$ , where the subscripts x, y, and z denote partial derivatives by the  $\mathbf{r}_1$  coordinates. By computation in coordinates, the  $dx_2$  component of the corresponding surface integrand of (4.26) differs from this by  $[(g_0A_3)_z + (g_0A_2)_y] dy_1 dz_1 - (g_0A_2)_x dz_1 dx_1 - (g_0A_3)_x dx_1 dy_1$ , which can be seen to be the  $dx_2$  component of  $d\star(g \wedge A)$ . Similar reasoning for the  $dy_2$  and  $dz_2$ components shows that the difference between Eqs. (4.22) and (4.26) is

$$i\omega \int_{\partial V} d\star (g \wedge A) \tag{4.28}$$

which vanishes since the integral of an exact differential over a closed region is zero, as can be verified by making use of the generalized Stokes theorem. Thus, the expression (4.22) for the electric field intensity in terms of the scalar Green function is equivalent to (4.26) in terms of the Helmholtz Green form.

Finally, I will show that the free space special case of the integral equation (4.1) can be derived from Eq. (4.26). This will complete the proof that for free space the results

of the previous chapter reduce to the usual expression (4.22) for the electric field in terms of the scalar Green function. Since  $E = i\omega A - d\phi$ , the integrand S of the surface contribution in Eq. (4.26) can be written as

$$S = g \wedge \star dE + \star dg \wedge E - (\star d \star g) \star E + \star dg \wedge d\phi + i\omega(\star g) \star d\star A - \phi \star d\star d\star g.$$
(4.29)

The first two terms of this expression are identical to the integrand of the surface contribution of Eq. (4.1). By rearranging Eq. (4.26),

$$E = i\omega\mu_0 \int_V g \wedge J + \int_V (\star d\star g)\rho/\epsilon_0 + \int_{\partial V} \left[g \wedge \star dE + \star dg \wedge E - (\star d\star g)\star E\right] + \int_{\partial V} S'$$
(4.30)

where  $S' = \star dg \wedge d\phi + i\omega(\star g) \star d\star A - \phi \star d\star d\star g$ . By using the Lorentz gauge  $\star d\star A = i\omega\epsilon_0\mu_0\phi$ , S' can be transformed into

$$S' = \star dg \wedge d\phi - \phi \star (d \star d \star + k_0^2)g. \tag{4.31}$$

The definition (4.23) shows that this is equal to

$$S' = \star dg \wedge d\phi - \phi \star (\star d \star dg - \delta I). \tag{4.32}$$

The first two terms of S' are equal to the exact form  $-d(\star \phi dg)$ , and so their integral over  $\partial V$  vanishes by Stokes theorem. The third term appears to lead to a singularity on  $\delta V$ , but the surface integral of S in the derivation of (4.26) originated from the volume integral of dS, so that the contribution of the third term in (4.32) is more precisely equal to

$$\int_{V} d(\phi \delta \star I) \tag{4.33}$$

which vanishes due to the identity

$$\int f(x)\frac{\partial}{\partial x}\delta(x-a)\,dx = -\frac{\partial f}{\partial x}(a). \tag{4.34}$$

Thus, the term containing S' vanishes and Eq. (4.30) simplifies to

$$E = i\omega\mu_0 \int_V g \wedge J + \int_V (\star d\star g)\rho/\epsilon_0 + \int_{\partial V} \left[g \wedge \star dE + \star dg \wedge E - (\star d\star g)\star E\right].$$
(4.35)

The surface integral term of this expression is equivalent to the Stratton-Chu formula [59].

Using the continuity equation  $dJ = i\omega\rho$ , Eq. (4.35) can be rewritten as

$$E = i\omega\mu_0 \int_V g\wedge J + \int_V (\star d\star g) dJ / (i\omega\epsilon_0) + \int_{\partial V} \left[g\wedge \star dE + \star dg\wedge E - (\star d\star g)\star E\right].$$
(4.36)

Integrating the second term by parts and using Ampere's law,

$$E = i\omega\mu_0 \int_V g\wedge J - \int_V d\star d\star g\wedge J/(i\omega\epsilon_0) + \int_{\partial V} \left[g\wedge \star dE + \star dg\wedge E + (\star d\star g)dH/(i\omega\epsilon_0)\right].$$
(4.37)

The volume integral terms show that the Green form for the electric field can be written as

$$G = \left(1 + \frac{1}{k_0^2} d\star d\star\right)g\tag{4.38}$$

which is the usual result [3, 59] for an isotropic, homogeneous medium.

Gauss's law for the electric field can be used to replace  $\rho/\epsilon_0$  with  $d\star E$  in Eq. (4.35), so that the expression becomes a volume integral equation,

$$E = i\omega\mu_0 \int_V g \wedge J + \int_V (\star d\star g) d\star E + \int_{\partial V} \left[g \wedge \star dE + \star dg \wedge E - (\star d\star g) \star E\right] \quad (4.39)$$

where the unknown field E appears under the volume integral on the right hand side. For the isotropic case, this reformulation is clearly not advantageous. By integrating the second term by parts, however, Eq. (4.39) can be written as

$$E = i\omega\mu_0 \int_V g \wedge J - \int_V \star d\star d\star g \wedge E + \int_{\partial V} \left[g \wedge \star dE + \star dg \wedge E\right]. \tag{4.40}$$

For a homogeneous, isotropic medium, the integral equation (4.1) reduces essentially to this expression. If the free space Helmholtz Green form g in (4.40) is replaced with  $\tilde{\star}_h \tilde{g}$ ,  $\star$ replaced with  $\tilde{\star}_h$  or  $\star_h$ , and the factors of  $\mu_0$  removed, then the equation becomes identical to (4.1). For the case of a medium with symmetric permeability tensor, the  $\tilde{\star}_h$  operator can be absorbed into the definition of  $\tilde{g}$  as was done in Sec. 3.3.2, and the correspondence between this expression and the general integral equation (4.1) becomes even closer. We have now obtained the purpose of this section, which is to demonstrate the connection between the usual free space expressions (4.20) and (4.21) and the integral equation derived in the previous chapter.

## 4.2.1 Plane Wave Solutions

In free space, plane wave solutions for the electric field correspond to poles of the Green form g. In spite of this, the second volume integral term of (4.40) remains finite even if E represents a plane wave and V is unbounded, due to the presence of the derivative operator  $\star d \star d \star$  and the constraint imposed by Gauss's law. This can be seen explicitly by considering a very simple example. If E is equal to  $E_0 e^{ik_0 z} dx$ , then the right–hand side of the integral equation (4.40) becomes

$$-E_0 \int_{V_1} dx_1 \, dy_1 \, dz_1 e^{ik_0 z} \left( dx_2 \frac{\partial^2 g_0}{\partial x_1^2} + dy_2 \frac{\partial^2 g_0}{\partial x_1 \partial y_1} + dz_2 \frac{\partial^2 g_0}{\partial x_1 \partial z_1} \right) \tag{4.41}$$

This integral can be evaluated as the inverse Fourier transform of the product of the transforms of  $e^{ikz}$  and  $g_0$ ,

$$E_{0}\frac{1}{8\pi^{3}}\int d\mathbf{k}\,e^{i\mathbf{k}\cdot\mathbf{r}_{2}}8\pi^{3}\delta(k_{z}-k_{0})\delta(k_{x})\delta(k_{y})\left(dx_{2}\frac{k_{x}^{2}}{k^{2}-k_{0}^{2}}+dy_{2}\frac{k_{x}k_{y}}{k^{2}-k_{0}^{2}}+dz_{2}\frac{k_{x}k_{z}}{k^{2}-k_{0}^{2}}\right).$$
(4.42)

After performing the  $k_z$  and  $k_y$  integrations, this becomes

$$E_0 \int dk_z \,\delta(k_x) \,dx_2 e^{ik_z z_2} \frac{k_x^2}{k_x^2} = E_0 e^{ik_0 z_2}. \tag{4.43}$$

so that the volume integration of (4.40) does yield E, as expected. Note that the integral in (4.40) becomes singular if E does not satisfy Gauss's law and the wavevector is not orthogonal to E.

For a general plane wave  $E_0e^{i\mathbf{l}\cdot\mathbf{r}}$ , where  $E_0 = E_1 dx + E_2 dy + E_3 dz$  is a constant 1-form, the inverse Fourier transform integral becomes

$$\frac{1}{8\pi^3} \int d\mathbf{k} \, e^{i\mathbf{k}\cdot\mathbf{r}_2} 8\pi^3 \delta(\mathbf{k}-\mathbf{l}) \frac{\mathbf{k} \, \, \Box E_0}{k^2 - k_0^2} \left(k_x \, dx_2 + k_y \, dy_2 + k_z \, dz_2\right) \tag{4.44}$$

so that I and  $E_0$  must be orthogonal in order for the integral to converge. Also, as above the integrations in the plane of the wavevector space perpendicular to I must be performed before the integration in the I direction. A similar computation can also be performed for a plane wave propagating in a biaxial medium.

#### 4.3 Singularity of the Helmholtz Green Form

Due to the singularity of  $g(\mathbf{r}_1, \mathbf{r}_2)$  at  $\mathbf{r}_1 = \mathbf{r}_2$ , the derivation of (4.1) in the previous chapter was not strictly correct. As shown by Yaghjian [53] for the isotropic case,

however, the final result is valid if the proper principal value interpretation for integrals involving second order partial derivatives of g is employed. The result is also valid if the expression can be placed into a form such that only first order derivatives of g are present [52].

The second order derivatives of  $\tilde{g}$  can be eliminated from (4.1) by integrating the second volume integral term by parts, so that the integral equation becomes

$$E = i\omega \int_{V} \tilde{g} \wedge \star_{h} J + \int_{V} \star_{h} d\tilde{g} \wedge d\star_{h} E + \int_{\partial V} \left[ \tilde{\star}_{h} \tilde{g} \wedge \tilde{\star}_{h} dE + \star_{h} d\tilde{\star}_{h} \tilde{g} \wedge E - (\star_{h} d\tilde{g}) \star_{h} E \right].$$

$$(4.45)$$

This is a generalization of the Stratton–Chu formula [59]. In a free space region containing no sources, the Stratton–Chu formula is a surface integral equation, since  $d\star_h E = \mu_0 \rho/\epsilon_0 = 0$ . For a complex medium, this generalization of the Stratton–Chu formula is a volume integral equation, since  $d\star_h E$  is not related to  $d\star_e E = \rho$  in any simple manner.

For an arbitrary fundamental solution g of the anisotropic Helmholtz equation, the surface contribution of (4.45) does not vanish. In order for the solution E given by (4.45) to be physically meaningful, it must satisfy a specified boundary condition on  $\partial V$ . If the electric field E and  $g(\mathbf{r}_1, \mathbf{r}_2)$  as a function of  $\mathbf{r}_1$  satisfy a boundary condition such that the surface contribution to (4.45) vanishes, then the result given by (4.45) will satisfy the same boundary condition as  $g(\mathbf{r}_1, \mathbf{r}_2)$  as a function of  $\mathbf{r}_2$ . As shown in Sec. 3.2.1, the first two terms of the surface contribution do not contribute if g and E satisfy magnetically conducting, electrically conducting, or radiation boundary conditions. Unfortunately, the term  $(\star_h d\tilde{g}) \star_h E$  in general does not vanish. In order to avoid the additional surface integral term, the integral equation (4.1) could be employed directly instead of (4.45). In order to do this, one must determine the correct principal value interpretation for the volume integral

$$\int_{V_1} \check{\star}_h d\star_h d\tilde{g}(\mathbf{r}_1, \mathbf{r}_2) \wedge E(\mathbf{r}_1).$$
(4.46)

Proper treatment of the integration is crucial, since E is in general nonzero over all of Vand so evaluation of the integral term at  $\mathbf{r}_2 = \mathbf{r}_1$  cannot be avoided. For free space, a nontrivial principle value interpretation is required for the volume integral terms of (4.37) only if the value of the electric field is desired at a location for which  $J \neq 0$ . I will determine the principal value interpretation for the case of a biaxial medium. In this case, Eq. (4.1) simplifies to (4.6). The Helmholtz Green form is a double  $1 \otimes 1$  form with diagonal elements  $e^{ik_{0i}r}/(4\pi r)$ . The volume integral term (4.46) becomes

$$\int_{V} \star d \star d \star g(\mathbf{r}_{1}, \mathbf{r}_{2}) \wedge E(\mathbf{r}_{1}).$$
(4.47)

I assume that E satisfies a Holder condition in the interior of V, so that loosely speaking, the value of E does not vary too much over any small region. With this condition, a principal value interpretation leading to a uniquely defined value for integrals of the form of (4.47) is known to exist [52]. The domain of the volume integration can be divided into two parts,  $V - V_{\delta}$  and  $V_{\delta}$ , where  $V_{\delta}$  contains the point  $\mathbf{r}_2$ . The volume integral of  $\star d \star d \star g \wedge E$  is then equal to

$$\int_{V-V_{\delta}} d\star d\star g \wedge \star E + \int_{V_{\delta}} d\star d\star g \wedge \star E.$$
(4.48)

Integrating the second term by parts and applying Stokes theorem yields

$$\int_{V-V_{\delta}} d\star d\star g \wedge \star E + \int_{S_{\delta}} \star d\star g \wedge \star E - \int_{V_{\delta}} \star d\star g \wedge d\star E$$
(4.49)

where  $S_{\delta}$  is the boundary of  $V_{\delta}$ . The first term represents the value which is obtained by numerical integration of Eq. (4.47) for the particular exclusion volume  $V_{\delta}$ . The second term represents a correction to this value such that the sum of the first two terms is independent of the choice of shape for  $V_{\delta}$ . The third term vanishes in the limit as the maximal dimension  $\delta$  of  $V_{\delta}$  becomes small, since  $\star d \star g$  has a singularity which is only of order  $1/r^2$ , where  $r = |\mathbf{r}_1 - \mathbf{r}_2|$ .

It remains to compute the limit of the second term of Eq. (4.49) as  $\delta \to 0$ . In the limit,

$$\begin{aligned} \star d \star g &= g_{1x} \, dx_2 + g_{2y} \, dy_2 + g_{3z} \, dz_2 \\ &= \left( ik_{01} r_x - \frac{r_x}{r} \right) \frac{e^{ik_{01}r}}{4\pi r} \, dx_2 + \left( ik_{02} r_y - \frac{r_y}{r} \right) \frac{e^{ik_{02}r}}{4\pi r} \, dy_2 + \left( ik_{03} r_z - \frac{r_z}{r} \right) \frac{e^{ik_{03}r}}{4\pi r} \, dz_2 \\ &\simeq -\frac{1}{4\pi r^2} (r_x \, dx_2 + r_y \, dy_2 + r_z \, dz_2) \\ &= \frac{d_2 r}{4\pi r^2} \end{aligned}$$

where the subscript on  $d_2r$  indicates that the exterior derivative of  $r = |\mathbf{r}_1 - \mathbf{r}_2|$  is with respect to the  $\mathbf{r}_2$  coordinates. By using this result, the surface integral term in (4.49) can be

written

$$\lim_{\delta \to 0} \int_{S_{\delta}} \frac{d_2 r}{4\pi r^2} \star E(\mathbf{r}_1).$$
(4.50)

The value of the integral becomes linear in components of E [52], so that using the double form

$$L(\mathbf{r}_2, \mathbf{r}_1) = -\lim_{\delta \to 0} \int_{S_\delta} \frac{d_2 r}{4\pi r^2} I$$
(4.51)

where I here is the unit double form  $dy_3 dz_3 dx_1 + dz_3 dx_3 dy_1 + dx_3 dy_3 dz_1$ , and the integration is over the  $\mathbf{r}_3$  coordinates. The pullbacks of the 2-form factors of I to  $S_{\delta}$  are components of the surface normal  $\hat{\mathbf{n}}$  of  $S_{\delta}$ , so that this result for L is equivalent to that obtained by Yaghjian [53] for free space. Yaghjian gives results for L corresponding to several commonly employed shapes for the exclusion volume.

In terms of this result for the double form L, the volume integral (4.47) is equal to

$$\lim_{\delta \to 0} \int_{V-V_{\delta}} \star d \star d \star g \wedge E - L \, \lrcorner E(\mathbf{r}_2) \tag{4.52}$$

where the interior product acts on the  $\mathbf{r}_1$  differentials of L and the  $\mathbf{r}_2$  differentials of E. The clumsiness of the coordinate dependencies in this term is due to the fact that L would more naturally have the delta function coefficient  $\delta(\mathbf{r}_1 - \mathbf{r}_2)$  and be integrated against  $E(\mathbf{r}_1)$  over V. I have chosen to mimic the standard dyadic treatment, for which coordinate dependence is somewhat ambiguous and expressions such as (4.52) appear natural. Inserting this result into the integral equation (4.40) gives

$$E = i\omega\mu_0 \int_V g \wedge J + \lim_{\delta \to 0} \int_{V-V_{\delta}} \star d\star d\star g \wedge E - L \, \lrcorner E + \int_{\partial V} \left(g \wedge \star dE + \star dg \wedge E\right).$$
(4.53)

A similar derivation can be performed for the more general homogeneous, anisotropic case. If the medium is magnetically anisotropic, then the components of L contain factors related to the value of the permeability tensor for the medium.

#### 4.4 Summary

In this chapter, I have discussed several issues related to the application of the electric field integral equation which was derived in the previous chapter. I have compared this integral equation to the usual equivalent source formulation used for electrically inhomogeneous or anisotropic media, and pointed out cases where the present integral equation

may be superior as a basis for computational methods. Sec. 4.2 explored the connection between the results of the previous chapter and the usual solution for the electric field in terms of the scalar Green function for an isotropic, homogeneous medium. The magnetic vector potential **A** and the electric potential  $\phi$  can be written in terms of the scalar Green function, leading to an expression for the electric field. An equivalent formulation of this result can be obtained using the free space Helmholtz Green form. Although the Helmholtz Green form and the scalar Green function are trivially related for an isotropic, homogeneous medium, it is interesting to note that the proof of the equivalence of the two formulations given in Sec. 4.2 was not trivial. Once the standard free space result is expressed in terms of the Helmholtz Green form, its connection with the integral equation of the previous chapter becomes clear. The general integral equation (4.1) is a direct generalization of the free space result.

In Sec. 4.3, a principal value interpretation of the integrals in Eq. (4.1) was obtained for the case of a biaxial medium. Such an interpretation is required in order to implement this integral equation as a numerical algorithm. An additional term linear in E which depends on the geometry of a specified exclusion volume must be combined with the numerical value of the integral in order to give a result which is independent of the type of limiting process chosen in the numerical evaluation of the integral. For a biaxial medium, this term is the same as that obtained by previous authors for the homogeneous, isotropic case.

#### Chapter 5

# GAUSSIAN BEAMS IN BIAXIAL MEDIA

#### 5.1 Introduction

Some electromagnetic problems in homogeneous, anisotropic media, such as the analysis of optical devices relying on anisotropic effects, can be reduced to the study of narrow beams. In this chapter, I treat Gaussian beam solutions in a biaxial medium for directions of propagation away from the two optical axes of the medium. Propagation of beams with wave vector along an optical axis behave in a singular manner, and the associated phenomenon of internal conical refraction will be treated in the next chapter.

In order to compute the Gaussian beam solutions, a parabolic expansion for the wave surface will be employed. Such an expansion has been used by many authors, including Čtyroký [60] to give an integral formula for the Fresnel diffraction of a narrow beam, and Moskvin *et al.* [17] to obtain the far field limit of the tensor Green function for a biaxial medium. As shown in Chap. 3, the Green form for a biaxial, nonmagnetic medium can be expressed easily in the wavevector representation. The physical space representation can only be obtained in certain limits. To obtain the far field limit of the Green form itself, the inverse Fourier transform of the tensor Green function can be evaluated using stationary phase [16, 17]. A similar approach is employed in this chapter. I express the product of the Green form for the electric field and an equivalent Gaussian current source in the wavevector representation, and employ a parabolic wave surface expansion to obtain a paraxial approximation for the inverse Fourier transform of the product. This gives the electric field corresponding to a Gaussian beam with waist at the location of the equivalent current source.

Other approaches to the study of narrow beams in anisotropic media include that of Shin and Felsen [61], who make use of the free space scalar Green function with a complex position vector. This yields an exact solution to Maxwell's laws which reduces to a Gaussian beam in the paraxial limit. Using this method, Shin and Felsen give analytic results for the beam solutions for the special case of a uniaxial medium. Fleck and Feit [62] derive a paraxial wave equation in order to obtain the Gaussian beam solutions for a uniaxial medium. Ermert [63] derives another type of paraxial wave equation and gives the associated beam solutions for a biaxial medium, but these are only valid if certain conditions are placed on the principal permittivities of the medium.

# 5.2 Spectral Decomposition of the Green Form

I begin with the wavevector representation (3.66) of the Green form for the electric field obtained in Chap. 3. Since a biaxial medium is magnetically isotropic, for convenience I scale the Green form by a factor of  $\mu_0$ , so that G becomes

$$G = \left[ -\mathbf{k}\mathbf{k}^{T} + \mathbf{k}^{T}\mathbf{k}\mathbf{I} - \omega^{2}\mu_{0}\overline{\overline{\epsilon}} \right]^{-1}.$$
 (5.1)

where  $\overline{\epsilon}$  is the real, symmetric permittivity tensor of the medium. By using the notation  $\mathbf{k} = k\hat{\mathbf{n}}$ , this can be rewritten as

$$G(\mathbf{k},\omega) = \left[k^2(\mathbf{I} - \hat{\mathbf{n}}\hat{\mathbf{n}}^T) - \omega^2 \mu_0 \overline{\overline{\epsilon}}\right]^{-1}.$$
(5.2)

As discussed in the previous chapter, the zeros of the denominator of the Green form lead to the Fresnel equation. When considered as a quadratic equation in  $k^2$ , the Fresnel equation has one zero solution and two nonzero solutions. These solutions define the wave surface for the medium. For each direction  $\hat{\mathbf{n}}$ , the two corresponding values of k represents the distance from the origin to the wave surface. Since there are two roots for each direction, the wave surface consists of two parts, the internal part and the external parts. The external and internal parts meet at four points, and these points are in the directions of the two optical axes of the medium. Reference [7] contains an illustration of the wave surface for a biaxial medium.

Since parabolic approximations for the wave surface must be found for both the internal and external parts, a more convenient representation for the Green form is a spectral decomposition, so that G is separated into terms corresponding to each of the of roots of the Fresnel equation individually. This spectral decomposition is derived by Lax and Nelson [16]. Following their treatment, I define right eigenvectors  $v_j$  and eigenvalues
$k_i^{-2}$  such that

$$\bar{\bar{\epsilon}}^{-1}(\mathbf{I} - \hat{\mathbf{n}}\hat{\mathbf{n}}^T)\mathbf{v}_j = \frac{\omega^2 \mu_0}{k_i^2} \mathbf{v}_j.$$
(5.3)

The left eigenvectors of  $\overline{\overline{\epsilon}}^{-1}(\mathbf{I} - \hat{\mathbf{n}}\hat{\mathbf{n}}^T)$  are easily seen to be equal to  $\mathbf{v}_j^T \overline{\overline{\epsilon}}$ . If the normalizations of the eigenvectors are chosen such that

$$\mathbf{v}_j^T \bar{\bar{\epsilon}} \mathbf{v}_j = 1 \tag{5.4}$$

then by the spectral decomposition theorem,

$$\left[s\mathbf{I} - \overline{\overline{\epsilon}}^{-1}(\mathbf{I} - \hat{\mathbf{n}}\hat{\mathbf{n}}^{T})\right]^{-1} = \sum_{j=1}^{3} \frac{\mathbf{v}_{j}\mathbf{v}_{j}^{T}\overline{\overline{\epsilon}}}{s - \frac{\omega^{2}\mu_{0}}{k_{j}^{2}}}$$
(5.5)

Setting  $s = \omega^2 \mu_0 / k^2$  and rearranging this expression gives

$$G = \sum_{j=1}^{3} \frac{\mathbf{v}_{j} \mathbf{v}_{j}^{T}}{\omega^{2} \mu_{0} \left(\frac{k^{2}}{k_{j}^{2}} - 1\right)}.$$
(5.6)

The definition (5.3) shows that the eigenvectors satisfy  $\left[k_j^2(\mathbf{I} - \hat{\mathbf{n}}\hat{\mathbf{n}}^T) - \omega^2 \mu_0 \overline{\overline{\epsilon}}\right] \mathbf{v}_j = 0$ . This is the Fourier transform of the wave equation satisfied by the electric field. The eigenvectors  $\mathbf{v}_j$  therefore correspond to plane wave solutions with wavenumbers equal to  $k_j$ . The  $k_j$  are solutions of the Fresnel equation det  $\left[k^2(\mathbf{I} - \hat{\mathbf{n}}\hat{\mathbf{n}}^T) - \omega^2 \mu_0 \overline{\overline{\epsilon}}\right] = 0$ . In the principal coordinate system of the permittivity tensor, for which  $\overline{\epsilon}$  has diagonal components  $\epsilon_i$ , the eigenvectors have components [16]

$$\mathbf{v}_{jk} = M \frac{\mathbf{n}_k}{k_j^2 - \omega^2 \mu_0 \epsilon_k} \tag{5.7}$$

where M is chosen such that the normalization (5.4) holds. This expression is singular for wavevector directions lying on the principal axes, but the eigenvectors can be obtained for such directions by taking the limit as the wavevector approaches a principal axis. A nonsingular representation for the eigenvectors  $\mathbf{v}_j$  in terms of the wavevector has also been obtained [17].

One of the  $\omega^2 \mu_0 / k_j$  is zero; the associated term of (5.8) represents the nonpropagating or static part of G. From (5.3), the eigenvector corresponding to the zero eigenvalue is proportional to  $\hat{\mathbf{n}}$ , so that G can be rewritten as

$$G = \sum_{j=1}^{2} \frac{\mathbf{v}_{j} \mathbf{v}_{j}^{T}}{\omega^{2} \mu_{0} \left(\frac{k^{2}}{k_{j}^{2}} - 1\right)} - \frac{\hat{\mathbf{n}}\hat{\mathbf{n}}^{T}}{\omega^{2} \mu_{0} \left(\hat{\mathbf{n}}^{T}\overline{\overline{\epsilon}}\hat{\mathbf{n}}\right)}$$
(5.8)

where the summed terms correspond to the external and internal parts of the wave surface and the right–most term corresponds to the static root of the wave equation.

### 5.3 Paraxial Approximation of the Green Form

A Gaussian beam consists of a narrow distribution of components with wavevectors spread about some central direction of wave propagation. The wave surface governs the propagation of the energy associated with each component. We must therefore expand the wave surface about the central direction of propagation of the beam. Let  $\mathbf{k}'$  denote the wavevector in the principal coordinates of the permittivity tensor. Let  $\mathbf{k}$  represent the wavevector in a rotated coordinate system such that the  $k_z$  axis is in the direction of the central wavevector of the beam. In the two coordinate systems, the components of the wavevector are related by

$$\mathbf{k}' = A\mathbf{k} \tag{5.9}$$

where A is the orthogonal matrix

$$A = \begin{bmatrix} \cos\theta\cos\phi & -\sin\phi & \sin\theta\cos\phi\\ \cos\theta\sin\phi & \cos\phi & \sin\theta\sin\phi\\ -\sin\theta & 0 & \cos\theta \end{bmatrix}.$$
 (5.10)

The angles  $\phi$  and  $\theta$  represent the direction of the central wavevector of the Gaussian beam in the principal coordinate system. The wave surface must then be expanded in terms of  $k_x$ and  $k_y$ , which give the deviation of the wavevector away from the central  $k_z$  direction.

In the principal coordinate system, the wave surface is given by the Fresnel equation  $F(k'_x, k'_y, k'_z) = 0$ , where

$$F(k'_x, k'_y, k'_z) = -k'^2 (k_{01}^2 k'^2_x + k_{02}^2 k'^2_y + k_{03}^2 k'^2_z) + k'^2_x k_{01}^2 (k_{02}^2 + k_{03}^2) + k'^2_y k_{02}^2 (k_{03}^2 + k_{01}^2) + k'^2_z k_{03}^2 (k_{01}^2 + k_{02}^2) - k_{01}^2 k_{02}^2 k_{03}^2$$
(5.11)

and  $k_{0i}^2 = \omega^2 \mu_0 \epsilon_i$ . By using the relationship  $\mathbf{k}' = k' \hat{\mathbf{n}}'$ , the Fresnel equation can be rewritten so that it is biquadratic in  $k'^2$ ,

$$F(k'_{x},k'_{y},k'_{z}) = -k'^{4}(k_{01}^{2}n'_{x}^{2} + k_{02}^{2}n'_{y}^{2} + k_{03}^{2}n'_{z}^{2}) + k'^{2} \left[n'_{x}k_{01}^{2}(k_{02}^{2} + k_{03}^{2}) + n'_{y}k_{02}^{2}(k_{03}^{2} + k_{01}^{2}) + n'_{z}k_{03}^{2}(k_{01}^{2} + k_{02}^{2})\right] - k_{01}^{2}k_{02}^{2}k_{03}^{2}.$$
 (5.12)

In the rotated coordinate system, the Fresnel equation  $F(A\mathbf{k})$  is no longer biquadratic, but the roots can still conveniently be expanded using the implicit function theorem. Let the function  $g(k_x, k_y)$  be defined such that  $F(A[k_x, k_y, g(k_x, k_y)]^T) = 0$ . The wave surface then can be written in the form  $k_z = g(k_x, k_y)$ . By expanding  $g(k_x, k_y)$  for small  $k_x$  and  $k_y$ , we obtain the parabolic approximation

$$k_z \simeq \alpha_j + g_1 k_x + g_2 k_y + \frac{g_{11}}{2} k_x^2 + g_{12} k_x k_y + \frac{g_{22}}{2} k_y^2 \equiv T_j$$
(5.13)

where  $\alpha_j$  is a solution to  $F(A[0, 0, \alpha_j]) = 0$  and j indexes the components of the wave surface, so that j = 1 corresponds to the external part and j = 2 to the internal part. The subscripts on g denote partial derivatives by  $k_x$  and  $k_y$ , and all derivatives of g are evaluated at  $k_x = 0, k_y = 0$ .

The Fresnel equation in the form of (5.12) can be written as

$$-k^4 P + k^2 Q - R = 0 (5.14)$$

where

$$P = k_{01}^2 \sin^2 \theta \cos^2 \phi + k_{02}^2 \sin^2 \theta \sin^2 \phi + k_{03}^2 \cos^2 \theta$$
  

$$Q = \sin^2 \theta \cos^2 \phi k_{01}^2 (k_{02}^2 + k_{03}^2) + \sin^2 \theta \sin^2 \phi k_{02}^2 (k_{03}^2 + k_{01}^2) + \cos^2 \theta k_{03}^2 (k_{01}^2 + k_{02}^2)$$
  

$$R = k_{01}^2 k_{02}^2 k_{03}^2.$$

The constant term of the wave surface expansion (5.13) is therefore

$$\alpha_j^2 = \frac{Q - (-1)^j \sqrt{Q^2 - 4PR}}{2P}.$$
(5.15)

The first order coefficients can be found using the implicit function theorem,

$$g_m = -\frac{F_m(A\mathbf{k})}{F_3(A\mathbf{k})} \tag{5.16}$$

where the subscripts denote partial derivatives by  $k_x$ ,  $k_y$ , and  $k_z$ . By applying the chain rule, this can be rewritten as

$$g_m = -\frac{A_{l'm}F_{l'}(A\mathbf{k})}{A_{l'3}F_{l'}(A\mathbf{k})}$$
(5.17)

where the index l' is summed and  $F_{l'}(A\mathbf{k})$  denotes the partial derivative of F by the l'th components of  $\mathbf{k}'$ . The derivatives of F with respect to the principal coordinates can be obtained from (5.11),

$$F_{l'}(k'_x,k'_y,k'_z) = -2k'_{l'}(k_{01}^2k'^2_x + k_{02}^2k'^2_y + k_{03}^2k'^2_z) - 2k_{0l'}^2k'_{l'}k'^2 + 2k'_{l'}k_{0l'}^2(k_0^2 - k_{0l'}^2).$$
(5.18)

The expansion coefficients are obtained by evaluating (5.17) at the point  $\mathbf{k} = (0, 0, \alpha_j)$ , which corresponds to  $\mathbf{k}' = \alpha_j (\sin \theta \cos \phi, \sin \theta \sin \phi, \cos \theta)$  in the principal coordinate system.

The second-order coefficients can be obtained by taking partial derivatives of the  $g_j$ . They are

$$g_{mn} = \frac{-F_{mn}F_3 + F_mF_{3n}}{F_3^2}$$
(5.19)

where

$$F_m = A_{l'm} F_{l'}(A\mathbf{k}) \tag{5.20}$$

$$F_{mn} = A_{l'm} A_{p'n} F_{l'p'}(A\mathbf{k}).$$
(5.21)

The second derivatives of F in the principal coordinate system are

$$F_{l'p'} = -4k'_{l'}k'_{p'}(k^2_{0p'} + k^2_{0l'}) - 2\delta_{l'p'} \left[ k^2_{01}k'^2_x + k^2_{02}k'^2_y + k^2_{03}k'^2_z + k^2_{0l'}k'^2 - k^2_{0l'}(k^2_0 - k^2_{0l}) \right].$$
(5.22)

When expanded, the expressions (5.17) and (5.19) contain numerous terms and can be simplified considerably, but this has been done adequately in Ref. [60].

I will give the coefficients explicitly for the special case of  $\phi = 0$ , for which the  $k_z$  axis lies in the x' - z' plane. In this plane, we have that  $\alpha_1^2$  is equal to the greater and  $\alpha_2^2$  is equal to the lesser of  $k_{02}^2$  and  $k_{01}^2 k_{03}^2/G$ , where  $G = k_{01}^2 \sin^2 \theta + k_{03}^2 \cos^2 \theta$ . For  $\alpha_j^2 = k_{02}^2$ , the coefficients of the expansion  $T_j$  become

$$g_{1} = g_{2} = 0$$

$$g_{11} = -\frac{1}{k_{02}}$$

$$g_{12} = 0$$

$$g_{22} = \frac{k_{02}(k_{02}^{2} - k_{01}^{2}\cos^{2}\theta - k_{03}^{2}\sin^{2}\theta)}{k_{01}^{2}k_{03}^{2} - k_{02}^{2}G}.$$
(5.23)

For  $\alpha_j^2 = k_{01}^2 k_{03}^2/G$ , the coefficients are

$$g_{1} = \frac{\sin \theta \cos \theta (k_{03}^{2} - k_{01}^{2})}{G}$$

$$g_{2} = 0$$

$$g_{11} = \frac{k_{01}k_{03}}{G^{3/2}} \left[ \frac{2(\sin \theta \cos \theta (k_{03}^{2} - k_{01}^{2}))^{2}}{k_{01}^{2}k_{03}^{2} - k_{02}^{2}G} - 1 \right]$$

$$g_{12} = 0$$

$$g_{22} = \frac{1}{k_{01}k_{03}\sqrt{G}} \left[ \frac{k_{02}^{2}(k_{01}^{2} + k_{03}^{2})G - k_{01}^{2}k_{03}^{2}G - k_{01}^{2}k_{02}^{2}k_{03}^{2}}{k_{01}^{2}k_{03}^{2} - k_{02}^{2}G} \right].$$
(5.24)

Note that  $g_{11}$  and  $g_{22}$  become singular at the two angles

$$\tan \theta = \pm \sqrt{\frac{k_{03}^2 (k_{02}^2 - k_{01}^2)}{k_{01}^2 (k_{03}^2 - k_{02}^2)}}$$
(5.25)

These angles correspond to the optical axes of the medium. In these directions the expansion (5.13) becomes invalid, and a more sophisticated treatment must be made, as will be done in Chap. 6.

## 5.4 Gaussian Beams

Using the expansion for the wave surface (5.13), we find the propagating part of the Green form G to be

$$G = \sum_{j=1}^{2} \frac{\alpha_{j}^{2} \mathbf{v}_{j} \mathbf{v}_{j}^{T}}{\omega^{2} \mu_{0} (k_{z}^{2} - T_{j}^{2})}$$
(5.26)

for small  $k_{\rho}$ . I place an equivalent surface current density  $J_s = \xi_0(\rho)p$  on the z = 0 plane, where

$$\xi_0(\rho) = \frac{2E_0}{\eta_0} e^{-\rho^2/w_0^2}$$
(5.27)

and the constants are the free space impedance  $\eta_0 = \sqrt{\mu_0/\epsilon_0}$  and the beam waist parameter  $w_0$ . p is a unit 1-form specifying the direction of the equivalent source. The electric field is then

$$E(\mathbf{r}_2) = i\omega\mu_0 \int G(\mathbf{r}_1, \mathbf{r}_2) \wedge J(\mathbf{r}_1)$$
(5.28)

where the integration is over the  $z_1 = 0$  plane.

The integration in physical space of (5.28) becomes a product in the wavevector representation. The electric field in physical space is therefore equal to the inverse Fourier transform of the product of the wavevector representations of the Green form and the equivalent source. The Fourier transform of the equivalent current is  $\xi_0(k_\rho)p = (2E_0/\eta_0)\pi w_0^2 e^{-w_0^2 k_\rho^2/4}p$ , so that the electric field is

$$E = \frac{i\omega\mu_0}{8\pi^3} \int d\mathbf{k} e^{i\mathbf{k}\cdot\mathbf{r}} \xi_0(k_\rho) \sum_{j=1}^2 \frac{\alpha_j^2 v_j(v_j \, \exists \, p)}{\omega^2 \mu_0(k_z^2 - T_j^2)}$$
(5.29)

where the  $v_j$  are 1-forms dual to the polarization vectors  $\mathbf{v}_j$  expressed in the rotated coordinate system. Integrating  $k_z$  by a countour closing in the upper-half plane gives

$$E = \frac{\alpha_j}{16\pi^2\omega} \int dk_x \, dk_y \, e^{ik_x x + ik_y y} \xi_0(k_\rho) \sum_{j=1}^2 e^{izT(\alpha_j)} v_j(v_j \, \lrcorner \, p)$$
(5.30)

for the outgoing solution. Substituting Eq. (5.13) for  $T_j$  and using the definition of  $\xi_0$  yields

$$E_j = \frac{\alpha_j w_0^2 E_0 e^{i\alpha_j z}}{8\pi\omega\eta_0} \int dk_x \, dk_y \, e^R v_j (v_j \, \lrcorner \, p).$$
(5.31)

where  $E_1$  represents the contribution due to the external sheet of the wave surface,  $E_2$  represents the internal contribution, and the exponent is

$$R = ik_x(x+g_1z) + ik_y(y+g_2z) + k_x^2\left(\frac{g_{11}}{2} - \frac{w_0^2}{4}\right) + k_xk_yg_{12} + k_y^2\left(\frac{g_{22}}{2} - \frac{w_0^2}{4}\right).$$

The remaining transverse integrations can be performed by rotating  $k_x$  and  $k_y$  to clear the  $k_x k_y$  term of the exponent in the integrand of (5.31). This yields

$$E_{j} = \frac{\alpha_{j}w_{0}^{2}E_{0}e^{i\alpha_{j}z}}{8\pi\omega\eta_{0}}\int dk_{x}\,dk_{y}\,\exp\left[ik_{x}C + ik_{y}D - k_{x}^{2}\left(\frac{w_{0}^{2}}{4} - A\right) - k_{y}^{2}\left(\frac{w_{0}^{2}}{4} - B\right)\right]v_{j}(v_{j}\,\lrcorner p)$$
(5.32)

where

$$A_{j} = \frac{1}{\alpha_{j}} \left( \frac{g_{11}}{2} \cos^{2} \gamma + g_{12} \sin \gamma \cos \gamma + \frac{g_{22}}{2} \sin^{2} \gamma \right)$$
$$B_{j} = \frac{1}{\alpha_{j}} \left( \frac{g_{11}}{2} \sin^{2} \gamma - g_{12} \sin \gamma \cos \gamma + \frac{g_{22}}{2} \cos^{2} \gamma \right)$$
$$C_{j} = (x + g_{1}z) \cos \gamma + (y + g_{2}z) \sin \gamma$$
$$D_{j} = -(x + g_{1}z) \sin \gamma + (y + g_{2}z) \cos \gamma$$
$$\cot 2\gamma = \frac{g_{11} - g_{22}}{2g_{12}}.$$

The contributions from the two parts of the wave surface are then equal to

$$E_j = \frac{\alpha_j w_0^2 E_0 e^{i\alpha_j z}}{2\omega\eta_0 \sqrt{w_0^2 - 4A_j} \sqrt{w_0^2 - 4B_j}} \exp\left[-\frac{C_j^2}{w_0^2 - 4A_j} - \frac{D_j^2}{w_0^2 - 4B_j}\right] v_j(v_j \,\lrcorner\, p).$$
(5.33)

If the wavevector lies in the y' = 0 plane, then this expression simplifies to

$$E_{j} = \frac{\alpha_{j}w_{0}^{2}E_{0}e^{i\alpha_{j}z}}{2\omega\eta_{0}\sqrt{w_{0}^{2} - 4g_{11}}\sqrt{w_{0}^{2} - 4g_{22}}} \exp\left[-\frac{(x + g_{1}z)^{2}}{w_{0}^{2} - 4g_{11}} - \frac{y^{2}}{w_{0}^{2} - 4g_{22}}\right]v_{j}(v_{j} \,\lrcorner\, p) \quad (5.34)$$

where the coefficients are given in Eq. (5.24).

In free space, the wave direction and the direction of the peak amplitude of a Gaussian beam coincide. For a biaxial medium, the coefficients appearing in the functions  $C_j$  and  $D_j$  are related to the angle of the ray direction away from the wave direction. The ray vector lies in the direction of the normal to the wave surface at the point  $(0, 0, \alpha_j)$ , and in general does not coincide with the wavevector. The functions  $C_j$  and  $D_j$  shift the peak of the Gaussian amplitude of the beam solution (5.33) so that it lies along the ray direction. Neglecting the complicated effects due to refraction at a face of a biaxial medium, the power contained in an incident beam splits into two parts, with the directions determined by normals to each sheet of the wave surface for the particular value of the wavevector of the incident beam.

If the wavevector coincides with one of the optical axes of the medium, the coefficients found in (5.34) become singular. As noted above, the treatment of this chapter is invalid for these directions. In the following chapter, the behavior of beams propagating along the optical axes are studied in detail.

#### Chapter 6

# **INTERNAL CONICAL REFRACTION**

#### 6.1 Introduction

The paraxial approximation for the Green form of the previous chapter breaks down if the direction of the wavevector about which the expansion is taken coincides with one of the four singular points of the wave surface for a biaxial medium. The singular points lie on two straight lines, which are known as optical axes or binormals. A narrow beam propagating along one of the optical axes of a biaxial medium spreads into a hollow cone. This phenomenon, internal conical refraction, was predicted by Hamilton in 1832 and observed shortly thereafter by Lloyd. A dark ring in the center of the circular intensity pattern produced by conical refraction was observed by Poggendorf in 1839 and later explained by Voigt. (These historical references and an elementary treatment of conical refraction are found in Born and Wolf [7].) Voigt's explanation of the Poggendorf dark ring was made more precise by Portigal and Burstein [64]. Lalor [65] and Juretschke [66] also reported methods for quantitative analysis of internal conical refraction. Schell and Bloembergen [67] further refined the work of Portigal and Burstein, achieving a result accurate to second order in angle away from the optical axis. Despite the improved accuracy, Schell and Bloembergen employed numerical integration in order to obtain some of the results given in the paper. Other theoretical treatments include that of Uhlmann [68], who proved the existence of the dark ring but did not examine the structure of the intensity pattern in detail. This chapter gives the treatment of internal conical refraction reported in Ref. [5].

Previous theoretical methods for obtaining the field intensity due to conical refraction amount to a two-dimensional stationary phase evaluation of an inverse Fourier transform integral for the refracted field intensity. This approximation for the field intensity can be understood geometrically, by considering the shape of the wave surface near an optical axis. The wave surface for a biaxial medium consists of an external and an internal sheet which meet in the directions of two optical axes. For wave directions away from the optical axes, power associated with the particular wavevector flows along two ray vectors, one normal to the external sheet of the wave surface and the other to the internal sheet. Near each singular point, the wave surface has the shape of a cone. Instead of two distinct normals, at a singular point the wave surface has a family of normal directions lying on another cone. Incident power propagates along this cone of normals. The contributions from nearby wavevectors on the internal and external sheets are shifted slightly to the inside and outside of the cone of refraction respectively, so that a dark ring appears in the center of the circular intensity pattern produced by conical refraction [67].

The treatment of conical refraction given in this chapter employs the wavevector representation of Lax and Nelson [16] for the Green function for the electric field which was used in the previous chapter. A conical expansion for the wave surface near an optical axis given by Moskvin *et al.* [17] yields a paraxial approximation for the Green function. The refracted fields can then be obtained by finding the inverse Fourier transform of the product of the Green function and the spectral representation of a Gaussian beam. I treat asymptotically an integration in azimuthal angle about the optical axis, and the remaining transverse integration can be evaluated analytically. The resulting simple characterization of the intensity pattern in terms of special functions is one of the primary contributions of this chapter to the theory of internal conical refraction. In order to demonstrate the validity of this approach, I have also performed numerical integrations for the field intensity at certain parameter values.

The results obtained in this way agree with the theoretical and experimental results of Schell and Bloembergen [67] for a 1 cm Aragonite sample, a 34  $\mu$ m beam waist, and a wavelength of .6328  $\mu$ m. For a 10 cm sample length, however, their theoretical results are qualitatively similar to the 1 cm pattern, whereas this treatment predicts secondary dark rings or fringes in the interior of the cone of refraction. I specify the parameter ranges for which this secondary oscillatory behavior of the intensity pattern should appear, and demonstrate that even allowing for large variation of the parameters the effect persists. These secondary dark rings have apparently not been predicted by past theoretical treatments, nor have experimental results been given for parameter values lying within this oscillatory regime. Measurements by Schell and Bloembergen [6] indicate the appearance of qualitatively similar secondary rings for conical refraction by an optically active medium. Oscillatory behavior of the intensity pattern has been predicted for conical refraction in gyrotropic media [69, 70], but the field has an Airy function dependence and is identically zero for certain distances from the cone of refraction. This behavior is qualitatively different from that reported here for biaxial media. Other related work includes that of Naida [71], who considers conical refraction in an inhomogeneous, weakly biaxial medium. Belskii [72] obtains transmission coefficients for a thin biaxial plate along the optical axes, and Belskii *et al.* [73] discuss the change in astigmatism of a Gaussian beam propagating along an optical axis. Khatkevich [74] shows that a conically refracted beam is not confined to a particular generator of the cone, and plane wave solutions near the optical axis are discussed by Alexandroff [75]. References [69, 70, 76] also investigate the application of conical refraction in gyrotropic media to beam focusing. A recent experimental measurement for conical refraction in KTP is found in Ref. [77].

### 6.2 **Propagation Along an Optical Axis**

To determine the electric field due to the internal conical refraction of a Gaussian beam, I begin with the decomposition of the Fourier transform of the Green form given in Chap. 5 for a biaxial medium. As in the previous chapter, the refracted field can be obtained from an equivalent current source at the focus of a Gaussian beam by an inverse Fourier transform. The two main problems are the determination of the proper paraxial expansion of the Green form for wave directions near an optical axis and the asymptotic evaluation of the inverse Fourier transform in the paraxial limit.

For a given wave vector direction  $\hat{\mathbf{n}}$ , the Fresnel equation is biquadratic in the length of the wave vector. The Fresnel equation therefore has two pairs of solutions, the members of each pair differing by a sign. The wave surface defined by these solutions consists of two sheets, one sheet for each pair of solutions. In four wave directions, the solutions become equal, so that the two sheets of the wave surface meet. At each of the singular points, the wave surface has the shape of a cone [17]. The singular points lie in pairs on two lines, which are the optical axes or binormals [8]. Due to the conical shape,

the parabolic expansion of the previous chapter breaks down, and some of the coefficients become infinite. A different type of expansion which respects the conical shape of the wave surfaces must be used near the singular points.

Let (x', y', z') be the principal coordinate system of the permittivity tensor. If the eigenvalues are ordered so that  $\epsilon_1 < \epsilon_2 < \epsilon_3$ , then by Eq. (5.25) the optical axes lie in the x'-z' plane at the angles

$$\tan \beta = \pm \sqrt{\frac{\epsilon_3(\epsilon_2 - \epsilon_1)}{\epsilon_1(\epsilon_3 - \epsilon_2)}}.$$
(6.1)

from the z' axis. Near these directions, the wave surface forms a cone. Let x, y, z be the rotated coordinates

$$x = x' \cos \beta - z' \sin \beta$$
  

$$y = y'$$
  

$$z = x' \sin \beta + z' \cos \beta$$
(6.2)

so that the z axis lies in the direction of one of the optical axes. The geometry is depicted in Fig. 6.1.



Figure 6.1: Geometry of internal conical refraction. The z direction is an optical axis. Normals to the wave surface at the singular point generate the cone of refraction.

In cylindrical coordinates associated with the rotated coordinate system, the wave surface has an expansion about  $k_{\rho} = 0$  of the form  $k_z = T_j$ , where [17]

$$T_j = k_2 + A[\cos\phi + (-1)^{j+1}]k_\rho - B_j(\phi)k_\rho^2$$
(6.3)

$$B_j(\phi) = B[1 + (-1)^j D\cos\phi][1 - (-1)^j E\cos\phi].$$
(6.4)

The j = 1 term corresponds to the external part of the wave surface and j = 2 to the inner part. The constants are

$$A = \frac{1}{2}\sqrt{\frac{(\epsilon_3 - \epsilon_2)(\epsilon_2 - \epsilon_1)}{\epsilon_1\epsilon_3}}$$
$$B = \frac{(\epsilon_3 + \epsilon_2)(\epsilon_2 + \epsilon_1)}{8\epsilon_1\epsilon_3k_{02}}$$
$$D = \frac{\epsilon_3 - \epsilon_2}{\epsilon_3 + \epsilon_2}$$
$$E = \frac{\epsilon_2 - \epsilon_1}{\epsilon_2 + \epsilon_1}$$

where  $k_{02} = \omega \sqrt{\epsilon_2 \mu_0}$ . The apex angle of the cone of refraction is 2A.

If we neglect the nonpropagating term, the tensor Green function for small  $k_{\rho}$  is given by Eq. (5.26) from the previous chapter,

$$G = \sum_{j=1}^{2} \frac{\epsilon_2 \mathbf{v}_j \mathbf{v}_j}{k_z^2 - T_j^2}$$
(6.5)

where the polarization vectors  $\mathbf{v}_j$  are expressed in the principal coordinate system. The vectors  $\mathbf{v}_j$  can be found from the electric flux density eigenvectors, which lie in the plane perpendicular to the *z* axis. The vectors  $\mathbf{D}_j$  corresponding to the  $\mathbf{v}_j$  are [64]

$$\mathbf{D}_{1} = \hat{\mathbf{x}} \cos \frac{\phi}{2} + \hat{\mathbf{y}} \sin \frac{\phi}{2}$$
$$\mathbf{D}_{2} = -\hat{\mathbf{x}} \sin \frac{\phi}{2} + \hat{\mathbf{y}} \cos \frac{\phi}{2}$$

where  $\phi$  is the azimuthal angle associated with the rotated coordinate system. The  $\mathbf{v}_j$  are proportional to  $\overline{\overline{\epsilon}}^{-1}\mathbf{D}_j$ , so that

$$\mathbf{v}_{1}' = N\left(\hat{\mathbf{x}}'\epsilon_{1}^{-1}\cos\beta\cos\frac{\phi}{2} + \hat{\mathbf{y}}'\epsilon_{2}^{-1}\sin\frac{\phi}{2} - \hat{\mathbf{z}}'\epsilon_{3}^{-1}\sin\beta\cos\frac{\phi}{2}\right)$$
(6.6)

with the normalization

$$N = \left(\epsilon_1^{-1}\cos^2\beta\cos^2\frac{\phi}{2} + \epsilon_2^{-1}\sin^2\frac{\phi}{2} + \epsilon_3^{-1}\sin^2\beta\cos^2\frac{\phi}{2}\right)^{-\frac{1}{2}}$$
(6.7)

The eigenvector  $\mathbf{v}_2$  is equal to  $\mathbf{v}_1(\phi + \pi)$ .

The eigenvectors  $\mathbf{v}_j$  can also be found from Eq. (5.7). The wavevector  $k\hat{\mathbf{n}} = (k_\rho \cos \phi, k_\rho \sin \phi, T_j)$  can be transformed into the principal coordinate system to yield  $\hat{\mathbf{n}}'$ . The  $k_j$  are given by  $k_j^2 = k_\rho^2 + T_j^2$ . Substituting these expressions into (5.7) and taking the limit as  $k_\rho$  goes to zero yields the result

$$\mathbf{v}_{j}' = M\left(\hat{\mathbf{x}}'\frac{k_{02}\sin\beta}{k_{02}^{2} - k_{01}^{2}} + \hat{\mathbf{y}}'\frac{\sin\phi}{2k_{02}A(\cos\phi + (-1)^{j+1})} + \hat{\mathbf{z}}'\frac{k_{02}\cos\beta}{k_{02}^{2} - k_{03}^{2}}\right)$$
(6.8)

where M is a normalization such that (5.4) holds. Transforming to the unprimed coordinate system,  $v_1$  can be shown to be parallel to

$$\hat{\mathbf{x}}\cos\left(\phi/2\right) + \hat{\mathbf{y}}\sin\left(\phi/2\right) + \hat{\mathbf{z}}2A\cos\left(\phi/2\right)$$
(6.9)

and  $\mathbf{v}_2$  to

$$-\hat{\mathbf{x}}\sin\left(\phi/2\right) + \hat{\mathbf{y}}\cos\left(\phi/2\right) - \hat{\mathbf{z}}^2 A \sin\left(\phi/2\right)$$
(6.10)

which is equivalent to the result obtained in Ref. [67].

As in the previous chapter, I place an equivalent Gaussian surface current  $J_s = \xi_0 p$  on the z = 0 plane at the waist of the beam, where

$$\xi_0 = \frac{2E_0}{\eta_2} e^{-\rho^2/w_0^2} \,\delta(z) \tag{6.11}$$

and the 1-form p specifies the polarization of the beam in the principal coordinate system. The constant  $w_0$  specifies the waist size of the beam at its focus and  $\eta_2$  is the wave impedance  $\sqrt{\mu_0/\epsilon_2}$ . The beam is assumed to be focused at the incident face of the medium. The electric field is then

$$E = \frac{i\omega\mu_0}{8\pi^3} \int d\mathbf{k} \, e^{i\mathbf{k}\cdot\mathbf{r}} G(\mathbf{k}) \, \lrcorner \xi_0(k_\rho) p \tag{6.12}$$

where  $\xi_0(k_\rho) = (2E_0/\eta_2)\pi w_0^2 e^{-w_0^2 k_\rho^2/4}$ . Integrating  $k_z$  by a contour closing in the upper half plane yields for z > 0,

$$E = -\frac{k_{02}}{8\pi^2\omega} \int k_{\rho} \, dk_{\rho} \, d\phi e^{ik_{\rho}(x\cos\phi + y\sin\phi)} \xi_0(k_{\rho}) \sum_j v_j(v_j \,\lrcorner\, p) e^{iT_j z} \tag{6.13}$$

where the  $v_j$  are 1-forms dual to the vectors  $\mathbf{v}_j$ . Substituting the expressions for  $\xi_0$  and  $T_j$  given above yields

$$E = -\frac{k_{02}w_0^2 E_0 e^{ik_{02}z}}{4\pi\omega\eta_2} \int k_\rho \, dk_\rho \, d\phi \sum_j \exp\left[ik_\rho (g(\phi) + (-1)^{j+1}Az) - k_\rho^2 \left(\frac{w_0^2}{4} + izB_j\right)\right] v_j(v_j \, \lrcorner p)$$
(6.14)

where the leading order phase as a function of  $\phi$  for both the external and internal terms is

$$g(\phi) = (x + Az)\cos\phi + y\sin\phi.$$
(6.15)

The phase is stationary at two angles; for each of the two terms one of the stationary points is nonphysical. The causal stationary points are

$$\cos\phi_j = (-1)^j \frac{(x+Az)}{\sqrt{(x+Az)^2 + y^2}}$$
(6.16)

where the signs are chosen by noting that  $\phi_j$  specifies the angle of the location on the external or internal sheet of the wave surface at which the surface normal is in the direction of the ray vector corresponding to the observation point (x, y, z). Integrating (6.13) by the method of stationary phase gives

$$E = -\frac{k_{02}w_0^2 E_0 e^{ik_{02}z}}{4\pi\omega\eta_2} \sum_j \sigma_j v_j (v_j \,\lrcorner\, p) \int k_\rho \, dk_\rho \left(\frac{2\pi}{k_\rho |g_j''|}\right)^{1/2} e^{ik_\rho b_j - k_\rho^2 a_j} \tag{6.17}$$

where

$$\sigma_j = \exp\left[i(-1)^{j+1}\frac{\pi}{4}\right],\tag{6.18}$$

$$g_j(\phi_j) = (-1)^{j+1} \sqrt{(x+Az)^2 + y^2} = -g_j''(\phi_j),$$
 (6.19)

$$a_j = \frac{w_0^2}{4} + iB_j(\phi_j)z,$$
(6.20)

$$b_j = (-1)^{j+1} [Az - \sqrt{(x+Az)^2 + y^2}],$$
 (6.21)

and the  $v_j$  are evaluated at the stationary point  $\phi_j$ .

For the stationary phase integration, the large parameter is  $k_{\rho}\sqrt{(x+Az)^2+y^2}$ , so that the stationary phase condition becomes invalid as  $k_{\rho}$  grows small. Due to the additional factor of  $k_{\rho}$  in the integrand, however, the complete integrand of (6.17) is quite accurate for all values of  $k_{\rho}$ . As the stationary phase condition becomes invalid, the factor of  $k_{\rho}$  causes the value of the integrand to grows small, so that the lack of stationarity of the phase contributes only a small error to the approximate value of the integral obtained below. This effect is related to the close agreement between  $xJ_0(x)$  and  $\sqrt{2x/\pi} \cos(x - \pi/4)$  for all values of x, including small x.

Since  $\phi_2 = \phi_1 + \pi$  and  $v_2 = v_1(\phi + \pi)$ , we have that  $v_2(\phi_2) = v_1(\phi_1)$ . From the form of Eq. (6.4) for  $B_j$ ,  $a_1 = a_2$ , and from Eq. (6.21),  $b_1 = -b_2$ . The two terms of the integral in (6.17) can then be combined, so that the electric field intensity becomes

$$E = -\frac{k_{02}w_0^2 E_0 e^{ik_{02}z}}{4\pi\omega\eta_2} v_1(v_1 \,\lrcorner\, p) \left(\frac{2\pi}{|g_1''|}\right)^{1/2} e^{i\pi/4} \int_{-\infty}^{\infty} \sqrt{k_{\rho}} \, dk_{\rho} e^{ik_{\rho}b_1 - k_{\rho}^2 a_1} \tag{6.22}$$

The remaining  $k_{\rho}$  integration can be evaluated exactly using the integral ([78], 3.462 #3)

$$\int_{-\infty}^{\infty} \sqrt{x} e^{ibx - ax^2} = \sqrt{\pi} e^{-i\pi/4} 2^{-1/4} a^{-3/4} e^{-b^2/(8a)} D_{1/2} \left(\frac{-b}{\sqrt{2a}}\right)$$
(6.23)

where  $D_{\nu}$  is the parabolic cylinder function. (A parabolic cylinder function of order 1/2 has also been obtained in connection with a different integral arising from conical refraction by a gyrotropic medium in a certain limit [69].)

Asymptotically, the function  $D_{1/2}(x)$  depends exponentially on the square of its argument. The expression (6.23) can be put into a more useful form in which this asymptotic dependence is extracted and combined with the existing exponential factor of  $e^{-b^2/(8a)}$ . This is done using the relationship ([78], 9.240)

$$D_{\nu}(x) = 2^{\nu/2} e^{-x^2/4} \left[ \frac{\sqrt{\pi}}{\Gamma(1/4)} {}_{1}F_1\left(-\frac{\nu}{2}, \frac{1}{2}, \frac{x^2}{2}\right) - \frac{x\sqrt{2\pi}}{\Gamma(-\nu/2)} {}_{1}F_1\left(\frac{1-\nu}{2}, \frac{3}{2}, \frac{x^2}{2}\right) \right]$$
(6.24)

for the parabolic cylinder function in terms of the hypergeometric function  $_1F_1$  and the expression ([78], 8.972 #1)

$$L_{n}^{\alpha}(x) = \frac{\Gamma(n+\alpha+1)}{\Gamma(n+1)\Gamma(1-\alpha)} {}_{1}F_{1}(-n,\alpha+1,x)$$
(6.25)

for the associated Laguerre function  $L_n^{\alpha}$  in terms of the hypergeometric function. These relationships, along with the identities  $\pi\sqrt{2} = \Gamma(1/4)\Gamma(3/4) = -\Gamma(-1/4)\Gamma(5/4)$ , yield the result

$$\frac{e^{-i\pi/4}}{\sqrt{2}a^{3/4}}e^{-b^2/(4a)} \left[\Gamma(1/2)\Gamma(5/4)L_{1/4}^{-1/2}\left(\frac{b^2}{4a}\right) - \frac{b}{\sqrt{a}}\Gamma(3/2)\Gamma(3/4)L_{-1/4}^{1/2}\left(\frac{b^2}{4a}\right)\right] \quad (6.26)$$

for the integral (6.23).

By using this result, the final  $k_{\rho}$  integration of Eq. (6.22) can be performed, so that the electric field intensity for the refracted Gaussian beam is

$$E = -\frac{\epsilon_2 w_0^2 E_0 e^{ik_{02}z}}{4\sqrt{\pi}[(x+Az)^2 + y^2]^{1/4} a_1^{3/4}} v_1(\phi_1) [v_1(\phi_1) \, \lrcorner \, p] \, F(a_1, b_1) \tag{6.27}$$

where

$$F(a,b) = e^{-b^2/(4a)} \left[ \Gamma(1/2)\Gamma(5/4)L_{1/4}^{-1/2}\left(\frac{b^2}{4a}\right) - \frac{b}{\sqrt{a}}\Gamma(3/2)\Gamma(3/4)L_{-1/4}^{1/2}\left(\frac{b^2}{4a}\right) \right].$$
(6.28)

For large z,  $a_1$  is approximately  $iB_1z$ , so that the asymptotic dependence of E is  $z^{-5/4}$ . This matches the result reported by Moskvin *et al.* [17] for the field due to a point source in directions lying on the cone of internal conic refraction.



Figure 6.2: A circular cross section of the cone of refraction.  $b_1$  is the distance from (x, y, z) to the cone in the x-y plane.

For wave directions not lying exactly on an optical axis, the direction of the eigenvector  $v_1$  deviates from the value given by Eq. (6.6), leading to error in (6.27) in addition to that introduced by the asymptotic evaluation of the azimuthal integration. The behavior of the  $v_j$  as a function of  $k_x$  and  $k_y$  can be obtained from Eq. (5.7). The first order correction to (6.6) linear in  $k_{\rho}$  leads to an integral of the form

$$\int_{-\infty}^{\infty} x^{3/2} e^{ibx - ax^2} = \frac{\sqrt{\pi} e^{i\pi/4}}{2\sqrt{2}a^{5/4}} e^{-b^2/(4a)} \left[ 2\Gamma(7/4) L_{3/4}^{-1/2} \left(\frac{b^2}{4a}\right) - \frac{b}{\sqrt{a}} \Gamma(5/4) L_{1/4}^{1/2} \left(\frac{b^2}{4a}\right) \right].$$
(6.29)

This yields a contribution to the field which falls off for large z as  $z^{-7/4}$ , compared to  $z^{-5/4}$  for the leading term.

For fixed z, the leading behavior of (6.27) at large distances from the cone of refraction in the x-y plane is the Gaussian term  $\exp\left[-b_1^2/(4a_1)\right]$ , where  $b_1$  is the distance from the circular section of the cone with radius Az and center at (-Az, 0, z) as shown in

Fig. 6.2. The term  $v_1(\phi_1) \, \lrcorner p$  modulates the intensity pattern as a function of angle around the cone in the *x*-*y* plane, as exhibited by Fig. 3 of Ref. [67]. The polarization of the electric field is parallel to the vector given by Eq. (6.10) with  $\phi = \phi_2$ , where  $\phi_2$  is by (6.16) equal to the angle around the cone of refraction as shown in Fig. 6.2.

#### 6.3 Numerical Validation and Interpretation of Results

Expression (6.27) is singular at the center of the cone of refraction, since the stationary phase condition used in obtaining (6.17) is invalid at that point. For points away from the center of the cone, (6.27) is quite accurate, as has been verified by numerical integration of (6.13). The  $k_{\rho}$  integral in (6.13) can be evaluated in terms of associated Laguerre functions or hypergeometric functions. The  $\phi$  integration is then performed numerically. Numerical results obtained in this manner for Aragonite ( $n_x = 1.530$ ,  $n_y = 1.680$ ,  $n_z = 1.685$  [67]), z = 10 cm, beam waist 34  $\mu$ m, vacuum wavelength .6328  $\mu$ m, and incident polarization in the x direction differ from the approximate expression (6.27) by less than two percent over most of the intensity pattern, as shown in Fig. 6.3.



Figure 6.3: Magnitude of  $E/E_0$  for Aragonite, z = 10 cm, beam waist 34  $\mu$ m, and wavelength .6328  $\mu$ m. The solid line is computed by numerical integration. On the same scale the percentage error is shown as a dashed line. Incident polarization is in the x direction. The cone of refraction intersects the x axis at x = -3.5 mm.



Figure 6.4: Magnitude of F(1 + iq, b). The local minimum along the b = 0 axis produces the dark ring in the intensity pattern of conical refraction. b > 0 corresponds to the interior of the cone and b < 0 to the exterior.



Figure 6.5: Magnitude of  $E/E_0$  for Aragonite, z = 1 cm,  $w_0 = 18 \ \mu\text{m}$ , and  $\lambda = .6328 \ \mu\text{m}$ . The singularity of (6.27) at the center of the cone of refraction appears at x = -.175 mm. Incident polarization is in the xdirection.

The oscillatory behavior of F(a, b) includes the well-known Poggendorf dark ring, but for certain values of the beam waist size, propagation distance, and permittivities of the biaxial medium, additional fringes appear on the inside of the cone, as shown by the plot of |F(1 + iq, b)| in Fig. 6.4. There are two conditions which must be met in order for the secondary oscillatory behavior of the field intensity pattern to appear. First,  $a_1$  must be such that  $F(a_1, b_1)$  is oscillatory as the distance  $b_1$  from the cone of refraction varies. Second, the radius Az of the cone of refraction must be greater than the distance of the first secondary fringe from the cone of refraction. The coefficients D and E are typically much less than unity, so that  $B_j \simeq B$ . We also need only consider the y = 0 section of the intensity pattern. The parameters  $a_1$ and  $b_1$  of Eq. (6.27) can be rescaled so that  $a_1 = 1 + i4Bz/w_0^2$  and  $b_1 = -2x/w_0$ . As can be verified by examining the behavior of F(1 + iq, b), the first of the above conditions then yields roughly

$$.33 < \frac{Bz}{w_0^2} < 3.8 \tag{6.30}$$

for an additional dark ring of at least ten percent variation. The second condition is satisfied if

$$\frac{Az}{w_0} > 2.7 \frac{Bz}{w_0^2} + 3.0 \tag{6.31}$$

These ranges are sufficiently large that for reasonable experimental values and parameter variations the oscillatory regime should easily be observed. Secondary dark rings should appear, for example, in the intensity pattern for an Aragonite crystal of length 1 cm, a wavelength of .6328  $\mu$ m, and a beam waist size of 18  $\mu$ m, as shown in Fig. 6.5. For these values,  $Az/w_0 = 9.7$  and  $Bz/w_0^2 = 1.0$ , so that both conditions (6.30) and (6.31) are satisfied even with an error of ten percent in the beam waist size, sample length, or permittivities of the medium. The experimental arrangement described by Schell and Bloembergen [67] would allow sufficient control of the parameters to remain well within the oscillatory regime of the intensity pattern.

For a 10 cm crystal length and a beam waist of  $34 \,\mu$ m, the theory given here also predicts fringing in the intensity pattern (Fig. 6.6). This does not match the numerical results of Schell and Bloembergen (Fig. 7c of Ref. [67]). Although the shift of the dark ring's minimum towards the interior of the cone and the larger amplitude of the inner peak agree qualitatively, the intensity pattern obtained by Schell and Bloembergen exhibits no additional fringing. It is also interesting to note that although the fringes appear on both sides of the primary dark ring when the external and internal contributions to the field are taken separately, the interference is such that the total field exhibits fringing only on the interior of the cone of refraction due to interference between the two contributions.



Figure 6.6: Same as Fig. 6.3, except that dashed lines are magnitudes of the internal and external contributions taken separately and the solid line is total intensity as given by Eq. (6.27).

## 6.4 Summary

The intensity pattern due to internal conical refraction of a narrow beam by a biaxial medium apparently has a more complicated structure than previously thought, since the theory given here predicts additional dark rings on the interior of the cone for certain values of the material parameters, beam waist size, wavelength, and propagation distance. The existence of the primary dark ring in the intensity pattern can be explained heuristically, as is done in Ref. [7], by noting that if a narrow beam is considered to be a sum of plane waves, the solid angle of the plane wave components reaching a thin circle near the dark ring vanishes as the thin circle approaches the dark ring. For the secondary dark rings, however, there is no such qualitative explanation.

For biaxial media, there apparently do not exist in the literature measurements of the intensity pattern for beam and material parameters for which secondary dark rings would be expected to appear. It seems desirable to further explore conical refraction experimentally. I have given a reasonable sample size and beamwidth for Aragonite at which the fringes should appear and indicated generally how beam and material parameters relate to the appearance of oscillatory behavior in the intensity pattern. Chapter 7

# CONCLUSION

The purpose of this dissertation is to provide new techniques for analysis of electromagnetic fields in anisotropic, inhomogeneous media. The primary results are a new formalism for tensor Green functions for anisotropic media, an integral equation generalizing to anisotropic, inhomogeneous media the standard isotropic solution method involving the Green function for the scalar Helmholtz equation, a new integral equation for the electric field for complex media, closed form expressions for the Gaussian beam wave solutions in biaxial media, and a precise analysis of internal conical refraction in biaxial media predicting the appearance of secondary dark rings in the associated intensity pattern.

The electric field integral equation obtained in Chap. 3 and discussed in Chap. 4 is of interest for two classes of media. The first is that of a homogeneous medium, for which the Helmholtz Green form can be obtained exactly. The integral equation then has a closed–form kernel, and may be useful as a basis for numerical methods. The second case is that of an isotropic, homogeneous medium, for which the Helmholtz Green form reduces to the Green function for the scalar scattering by the same medium. To support possible applications of this integral equation, I show how it reduces to standard results for the isotropic, homogeneous special case, give principal value interpretations for the integral involving highly singular derivatives of the Helmholtz Green form, and compare the equation to the usual integral equation method for electrically anisotropic and inhomogeneous media.

Important physical problems involving propagation in anisotropic media can be solved by a knowledge of the narrow beam solutions for the medium, including the analysis of optical devices which rely on anisotropic media. Since a biaxial medium is easier to analyze than more general media and is encountered commonly in applications, for the study of beams I restrict study to this special type of medium. For an unbounded biaxial medium, Chap. 5 gives the Gaussian beam solutions for all directions in the medium except those which are near one of the optical axes. The phenomenon of internal conical refraction of beams propagating along these singular directions has been analyzed before, but the more precise analysis of Chap. 6 predicts a new effect, the existence of secondary dark rings concentric to the well–known Poggendorf dark ring in the intensity pattern of the refracted beam. I provide a numerical validations of the result and specify ranges of material parameters which should be suitable for experimental verification of the effect.

In addition to the presentation of the results noted above, this dissertation is intended to help establish the calculus of differential forms as a tool for applied electromagnetics problems. Results supporting this aim include the development of a geometrical meaning for field quantities expressed as differential forms in Chap. 2 and Appendix B. Appendix A gives the derivation of a new representation for electromagnetic boundary conditions, which is shown in Chap. 2 to have a clear geometrical meaning. In Chap. 3, the Hodge star operator is extended to allow its use for characterization of complex media with nonsymmetric permeability or permittivity tensors, and important theorems and definitions such as that of the Laplace–de Rham operator are generalized to the case of a nonsymmetric star operator. If one employs vectors and dyads, the derivation of key results in this dissertation is hindered by the necessity of separately proving many special cases of required identities. These results extend the utility of the calculus of differential forms as a formalism for the development of new theoretical methods.

### 7.1 Further Research

It is hoped that new directions of research on the electromagnetics of complex media will arise from this work, either through direct applications of the results presented here or through new applications of the general approach and formalism for Green function theory. There are a number of avenues for further work in extending the methods developed in this dissertation. These include:

• Use of the electric field or Green form integral equation as a basis for numerical methods of analyzing propagation in anisotropic, homogeneous media or isotropic, inhomogeneous media for which the Helmholtz Green form can be represented exactly;

- Determination of solutions for the Helmholtz Green form for bounded homogeneous media of various shapes which satisfy specified boundary conditions;
- Determination of other exact solutions for the Helmholtz Green form; types of media for which this may be possible include complex media with a certain symmetry, such as an inhomogeneous, anisotropic medium with radially symmetric permittivity profile;
- Investigation of the Helmholtz Green form and the associated derivative operator for the case of a magnetically inhomogeneous medium;
- Further exploration of the Neumann series solution for the electric field, since there may be types of media for which the series can be resummed in physical space or the terms represented generally using special functions, leading to new field solutions for the particular medium;
- Use of the integral equation for the case of an isotropic, electrically inhomogeneous medium in seeking analytical, asymptotic or numerical results for electromagnetic scattering from a knowledge of the scalar Green function, since the integral equation links scalar scattering with full vector scattering for the same medium;
- Experimental verification of the secondary dark rings predicted by the results of Chap. 6 for the intensity pattern of internal conical refraction, and a theoretical analysis of the relationship between the secondary fringes predicted by this theory and those that appear in the intensity pattern for an optically active medium [6].

In addition, there are likely many unexplored applications of the calculus of differential forms in applied electromagnetics other than the theory of Green functions. The strengths of differential forms are most evident when one is deriving coordinate–free expressions, rather than solving a problem for a specific source or boundary condition in a particular coordinate system. This makes the calculus of differential forms ideal as a tool for seeking new theoretical approaches. Differential forms also provide a link between electromagnetics and a large body of pure mathematical theory, and methods from these fields might be applied to the problems of applied electromagnetics. I hope that this work on the theory of

Green functions will provide not only numerical and analytical approaches to the solution of problems involving electromagnetic propagation in complex media but also a foundation for further theoretical developments in other areas of electromagnetics through the use of the calculus of differential forms.

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## Appendix A

# **BOUNDARY CONDITIONS USING DIFFERENTIAL FORMS**

In this appendix I derive a new formulation for the boundary conditions at a discontinuity in the electromagnetic field using differential forms [2]. Thirring [12] and Burke [30, 44] treat boundary conditions using the calculus of differential forms. Thirring's approach is similar to that given here, but his expressions for boundary sources include a singularity in the direction normal to the boundary. The formulation given here is more closely analogous to the usual vector boundary conditions than those given by Burke and Thirring. As discussed in Sec. 2.4.5, the boundary conditions derived in this appendix have intuitive geometric interpretations.

## A.1 Derivation

In this section I derive an expression for sources on a boundary where a field is discontinuous. The boundary condition associated with a general field equation can be given in convenient form using an operator which projects a form to its component with surfaces orthogonal to the boundary. The specializations to magnetic field intensity and electric flux density are discussed further in Chap. 2.

## A.1.1 Representing Surfaces With 1-forms

If the continuous and differentiable function  $f(x_1, ..., x_n)$  vanishes (or is constant) along a boundary, then when interpreted graphically the 1-form df has a surface that lies on the boundary. This 1-form can be used to express boundary conditions for fields near a boundary. The surface of a paraboloid reflector antenna, for example, is given by  $-x^2 - y^2 + az = 0$ , so that the unnormalized boundary 1-form is -2x dx - 2y dy + a dz. A rough surface can be described by -h(x, y)+z = 0, giving the boundary 1-form -dh+dz. For the remainder of this appendix I will use the notation

$$n = \frac{df}{|df|} = \frac{df}{\sqrt{df \,\lrcorner df}} \tag{A.1}$$

where  $\exists$  is the interior product. The 1-form *n* is dual to the usual surface normal vector  $\hat{n}$ .

## A.1.2 The Boundary Projection Operator

In this section, I derive a general expression for the boundary conditions satisfied by a field which is related to a source and another nonsingular field by exterior differentiation. Let  $\alpha$  be a *p*-form with p < n (where *n* is the dimension of space) which represents a field with a (p + 1)-form  $\beta$  as a source, so that

$$d\alpha = \gamma + \beta \tag{A.2}$$

where  $\gamma$  is nonsingular. Let f = 0 represent a boundary, where f is differentiable and vanishes only along the boundary. Let  $\alpha$  equal  $\alpha_1$  for f > 0 and  $\alpha_2$  for f < 0.

We can write  $\alpha = (\alpha_1 - \alpha_2)\theta(f) + \alpha_2$ , where  $\theta$  is the unit step function. Then

$$\gamma + \beta = d\{(\alpha_1 - \alpha_2)\theta(f) + \alpha_2\}$$
  
=  $\delta(f)df \wedge (\alpha_1 - \alpha_2) + \theta(f)d(\alpha_1 - \alpha_2) + d\alpha_2.$  (A.3)  
=  $\tilde{\delta}(f)n \wedge (\alpha_1 - \alpha_2) + \theta(f)d(\alpha_1 - \alpha_2) + d\alpha_2$ 

where  $\delta$  is the Dirac delta function and  $\tilde{\delta}(f)$  is  $\delta(x^1 - x_0^1) \cdots \delta(x^n - x_0^n)$  such that the point  $(x_0^1, ..., x_0^n)$  lies on the boundary and  $\delta(f) = \tilde{\delta}(f)/\sqrt{df \, {}_{\!\!\!\!\!} df}$ . The singular parts of both sides of (A.3) must be equal, so that

$$\beta' = \tilde{\delta}(f)n \wedge (\alpha_1 - \alpha_2) \tag{A.4}$$

where  $\beta'$  is the singular part of  $\beta$ , representing the boundary source along f = 0. Since the source  $\beta'$  is confined to the boundary, it can be written [12]

$$\beta' = \delta(f)n \wedge \beta_s \tag{A.5}$$

where  $\beta_s$  is a *p*-form, the restriction of  $\beta'$  to the boundary. Integrating (A.4) and (A.5) over a small region containing the boundary shows that the equality

$$n \wedge \beta_s = n \wedge (\alpha_1 - \alpha_2). \tag{A.6}$$

must hold on the boundary. The interior product distributes over the exterior product according to the relationship

$$\alpha \,\lrcorner (\beta \wedge \gamma) = (\alpha \,\lrcorner \beta) \wedge \gamma + (-1)^p \beta \wedge (\alpha \,\lrcorner \gamma) \tag{A.7}$$

where p is the degree of  $\beta$  and  $\alpha$  is a 1-form. Taking the interior product of both sides of (A.6) with n and applying the identity (A.7) yields

$$n \lrcorner (n \land (\alpha_1 - \alpha_2)) = n \lrcorner (n \land \beta_s)$$
$$= (n \lrcorner n) \land \beta_s - n \land (n \lrcorner \beta_s).$$
(A.8)

By definition, we have that  $n \, \exists \, n = 1$ . Since the source  $\beta_s$  is confined to the boundary,  $n \, \exists \, \beta_s = 0$ . By interpreting this graphically, we see that the surfaces of  $\beta_s$  must be perpendicular to the boundary, so that  $\beta_s$  can contain no factor of n. Applying this to (A.8), we have

$$\beta_s = n \, \lrcorner \left( n \land \left( \alpha_1 - \alpha_2 \right) \right) \tag{A.9}$$

which is the central result of this appendix. Note that the source may be represented by a twisted form, as defined and discussed in detail in Ref. [2]. In this case, an orientation for  $\beta_s$  must be specified. In practice, the distinction between twisted and nontwisted forms can be ignored and the sense of the current or charge represented by  $\beta_s$  obtained precisely in a straighforward manner, as explained in the following section.

The operator  $n \, \lrcorner n \land$  can be interpreted as a boundary projection operator. Geometrically, its action on a differential form is to remove the component of the form with surfaces parallel to the boundary. The boundary projection of a 1-form has surfaces perpendicular to the boundary. The boundary projection of a 2-form has tubes perpendicular to the surface at every point.

Since the boundary condition derived above applies to any law of the form of Eq. (A.2), Maxwell's laws (2.4) lead to

$$n \sqcup (n \land (E_1 - E_2)) = 0$$
  

$$n \sqcup (n \land (H_1 - H_2)) = J_s$$
  

$$n \sqcup (n \land (D_1 - D_2)) = \rho_s$$
  

$$n \sqcup (n \land (B_1 - B_2)) = 0$$
  
(A.10)

where  $J_s$  is the surface current 1-form and  $\rho_s$  is the surface charge 2-form. In four-space we have dF = 0 and dG = j, where  $F = B + E \wedge dt$ ,  $G = D - H \wedge dt$  and  $j = \rho - J \wedge dt$ . All four boundary conditions can be expressed as

$$n \lrcorner (n \land (F_1 - F_2)) = 0$$
  
$$n \lrcorner (n \land (G_1 - G_2)) = j_s$$
(A.11)

where  $j_s = \rho_s - J_s \wedge dt$ .

## A.1.3 Orientation of Sources

The direction of flow of the surface current represented by  $J_s$  and the sign of the surface charge represented by  $\rho_s$  cannot be obtained directly from the forms  $J_s$  and  $\rho_s$ alone. If the labels 1 and 2 of the two sides of a boundary are interchanged, the signs of  $J_s$ and  $\rho_s$  also change. The signs of the vector surface current density  $\mathbf{J}_s$  and the scalar charge density  $q_s$  do not change. If the quantities  $\mathbf{J}_s$  or  $q_s$  are integrated to yield total current or charge, however, one must choose a differential path length or differential surface element. There are two possible signs for these differential elements. This additional sign is already present in the differential forms  $J_s$  and  $\rho_s$ .

Although the sign change with respect to the labeling of regions makes the sense of the sources represented by  $J_s$  and  $\rho_s$  more difficult to specify precisely than is the case with the vector boundary sources, the differential forms lead to simpler integral expressions for total current and charge. For many electromagnetic quantities, the vector representation and the representation as a differential form are duals, so that their components differ only by metrical coefficients. This is not the case for the surface current and charge density forms yielded by the boundary projection operator. The integral of the surface current density  $J_s$  over a path should yield the total current through the path. The 1-form  $J_s$  as obtained using the boundary projection operator satisfies this definition:

$$I = \int_P J_s \tag{A.12}$$

where P is a path. The sense of I is with respect to the direction of the 2-form  $n \wedge s$ , where s is the 1-form dual to the tangent vector s of P (so that s is a 1-form with surfaces
perpendicular to the path P and oriented in the direction of integration). The total current given by (2.6) in terms of the vector surface current density is less natural than (A.12).

The total charge on an area A of a boundary with surface charge  $\rho_s$  is

$$Q = \int_{A} \rho_s. \tag{A.13}$$

In order to obtain the proper sign for Q, the orientation of A must be such that a 2-form  $\omega$  which satisfies  $n \wedge \omega = \Omega$  also satisfies  $\int_A \omega > 0$ , where  $\Omega$  is the standard volume element, dx dy dz in rectangular coordinates. The sign of the charge represented by  $\rho_s$  can also be found by computing  $(n \wedge \rho_s)/\Omega$ . This complication in specifying the sense of Q is actually present when dealing with the scalar surface charge density  $q_s$  as well, since one must choose the area element dS and orientation of A in  $Q = \int_A q_s dS$  such that  $\int_A dS$  is positive in order to obtain the correct total charge.

# A.2 Boundary Decomposition of Forms

The boundary conditions of the previous section appear to depend on a metric, since the metric-dependent interior product of differential forms is employed. The boundary projection operator  $n \, \lrcorner n \land$ , however, is in a sense extraneous. The 1-form  $H_1 - H_2$ , for example, yields the same result as the surface current  $J_s = n \, \lrcorner (n \land (H_1 - H_2))$  for integration over any path lying in the boundary. The forms  $J_s$  and  $H_1 - H_2$  are equivalent in that they have the same restriction to the boundary. The boundary projection operator simply removes the component of the form with surfaces parallel to the boundary with respect to the metric. The metric has no effect on the relationship between the field discontinuity and boundary source.

A metric independent type of boundary decomposition can be defined. Let f be a function vanishing along a boundary as above. Let v be a an arbitrary vector field. The interior product expands across the exterior product according to

$$\mathbf{v} \lrcorner (\alpha \land \omega) = (\mathbf{v} \lrcorner \alpha) \land \omega + (-1)^p \alpha \land (\mathbf{v} \lrcorner \omega)$$
(A.14)

where p is the degree of  $\alpha$ . Recall that the interior product of a vector and a form represents the contraction of the vector with the leftmost index of the form, and is metric independent. If  $\alpha = df$ , then by rearranging (A.14) we have

$$(\mathbf{v} \,\lrcorner\, df)\omega = \mathbf{v} \,\lrcorner\, (df \wedge \omega) + df \wedge (\mathbf{v} \,\lrcorner\, \omega). \tag{A.15}$$

Since the term  $df \wedge (\mathbf{v} \sqcup \omega)$  contains a factor of df, its integral over any region lying in the boundary f = 0 must vanish, as can be verified by integration by parts. The interior product of  $\mathbf{v}$  with the term  $\mathbf{v} \sqcup (df \wedge \omega)$  vanishes by the antisymmetry of the tensor  $df \wedge \omega$ . Thus, Eq. (A.15) decomposes  $(\mathbf{v} \sqcup df)\omega$  into two parts, one which has zero contraction with  $\mathbf{v}$  and another which integrates to zero over any region confined to the boundary. Furthermore, if the vector  $\mathbf{v}$  is chosen such that  $\mathbf{v} \sqcup df = 1$ , then  $\mathbf{v} \sqcup df \wedge and df \wedge \mathbf{v} \sqcup$  are both projections.

If the vector  $\mathbf{v}$  is related to df by a metric, then the term  $\mathbf{v} \,\lrcorner\, (df \land \omega)$  is orthogonal to the boundary f = 0 in a metrical sense. In the previous section, the use of  $n \lrcorner$  instead of  $\mathbf{v} \lrcorner$  is equivalent to obtaining  $\mathbf{v}$  from df/|df| by raising its index using a metric.

If the boundary is sufficiently smooth, then there is a local coordinate system  $x_1, \ldots, x_n$  for which  $f = x_1$ . If the vector **v** is chosen to be  $\hat{\mathbf{x}}_1$  (or  $\partial_{x_1}$  in the notation of differential geometry), then the first term of the decomposition (A.15) consists of all terms of  $\omega$  which do not contain a factor of  $dx_1$ . This part of  $\omega$  is the restriction of  $\omega$  to the boundary, since it is equal to the pullback of  $\omega$  to the boundary by the function  $(0, x_2, \ldots, x_n)$ . The second term of the decomposition includes the remaining terms of  $\omega$  which contain a factor of  $dx_1$ .

It is interesting to compare Eq. (A.15) to the definition of the wave operator  $\Delta$ . For a constant metric the definition of the Laplace–de Rham operator can be written as [3]

$$(d \,\lrcorner d)\omega = d \,\lrcorner (d \wedge \omega) + d \wedge (d \,\lrcorner \omega) \tag{A.16}$$

where the interior product of the exterior derivative with another quantity is defined by using the metric formally to convert the operator  $d = \partial/\partial_{x_1} dx_1 + \cdots + \partial/\partial_{x_n} dx_n$  from a 1-form to a vector and then contracting this vector with the first index of the second factor of the interior product. The spatial Fourier transform of (A.16) is identical to (A.15) with v equal to the wavevector k and df replaced with the dual 1-form k.

Finally, I note that Burke's formulation of boundary conditions provide an elegant alternative proof of the result (A.9) [30]. In integral form, Eq. (A.2) is

$$\oint_{\partial M} \alpha = \int_{M} \gamma + \int_{M} \beta \tag{A.17}$$

As the region M approaches  $M \cap B$ , where B is the boundary f = 0, the right-hand side approaches

$$\int_{M} \gamma + \int_{M} \beta = \int_{M \cap B} \beta_s \tag{A.18}$$

where the  $\gamma$  term drops out since  $\gamma$  is nonsingular and so its integral vanishes as M loses its dimension in the direction perpendicular to B. In the same limit, the left–hand side reduces to

$$\oint_{\partial M} \alpha = \int_{\partial M_1} \alpha + \int_{\partial M_2} \alpha \tag{A.19}$$

where  $M_1$  is the part of M on the f < 0 side of the boundary and  $M_2$  is the part on the f > 0 side. This in turn becomes

$$\int_{\partial M_1} \alpha + \int_{\partial M_2} \alpha = \int_{M \cap B} (p_1^* \alpha + p_2^* \alpha)$$
(A.20)

where  $p_1$  is a mapping from the f > 0 side of the boundary to the boundary,  $p_2$  is a mapping from the f < 0 side of the boundary to the boundary, and the superscript \* represents the pullback operation. The integrand on the right is given the symbol  $[\alpha]$  by Burke. By combining Eqs. (A.18) and (A.20), we have that

$$\int_{M \cap B} [\alpha] = \int_{M \cap B} \beta_s \tag{A.21}$$

Since M can be chosen to be arbitrarily small, the integrands on both sides of this expression must be equal if  $\alpha$  and  $\beta_s$  are sufficiently regular. The boundary condition on  $\alpha$  can thus be written

$$[\alpha] = \beta_s. \tag{A.22}$$

By the above discussion, the pullback of  $\mathbf{v} \,\lrcorner (df \land \alpha)$  to the boundary is equivalent to  $[\alpha]$ . Furthermore, in a coordinate system  $x_1, \ldots, x_n$  such that  $f = x_1$ , then if  $\mathbf{v} = \hat{\mathbf{x}}_1$ , the form  $\mathbf{v} \,\lrcorner (df \land \alpha)$  is equal to  $[\alpha]$  expressed in the same coordinate system.

# Appendix B

# TEACHING ELECTROMAGNETIC FIELD THEORY USING DIFFERENTIAL FORMS

The material in this appendix is taken from Ref. [1]. It includes an elementary introduction to electromagnetic field theory using differential forms, simple computational examples, and a summary of the pedagogical advantages of differential forms. The primary contribution of this appendix is to extend the geometric viewpoint advanced in Refs. [23] and [30], providing a new viewpoint on the quantities and physical principles of electromagnetic field theory. This viewpoint is a valuable tool in both teaching and research.

# **B.1** Introduction

Certain questions are often asked by students of electromagnetic (EM) field theory: Why does one need both field intensity and flux density to describe a single field? How does one visualize the curl operation? Is there some way to make Ampere's law or Faraday's law as physically intuitive as Gauss's law? The Stokes theorem and the divergence theorem seem vaguely similar; do they have a deeper connection? Because of difficulty with concepts related to these questions, some students leave introductory courses lacking a real understanding of the physics of electromagnetics. Interestingly, none of these concepts are intrinsically more difficult than other aspects of EM theory; rather, they are unclear because of the limitations of the mathematical language traditionally used to teach electromagnetics: vector analysis. In this appendix, we show that the calculus of differential forms clarifies these and other fundamental principles of electromagnetic field theory.

The use of the calculus of differential forms in electromagnetics has been explored in several important papers and texts, including Misner, Thorne, and Wheeler [23], Deschamps [33], and Burke [30]. These works note some of the advantages of the use of differential forms in EM theory. Misner *et al.* and Burke treat the graphical representation of forms and operations on forms, as well as other aspects of the application of forms to electromagnetics. Deschamps was among the first to advocate the use of forms in teaching engineering electromagnetics.

Existing treatments of differential forms in EM theory either target an advanced audience or are not intended to provide a complete exposition of the pedagogical advantages of differential forms. This appendix presents the topic on an undergraduate level and emphasizes the benefits of differential forms in teaching introductory electromagnetics, especially graphical representations of forms and operators. The calculus of differential forms and principles of EM theory are introduced in parallel, much as would be done in a beginning EM course. We present concrete visual pictures of the various field quantities, Maxwell's laws, and boundary conditions. The aim of this appendix is to demonstrate that differential forms are an attractive and viable alternative to vector analysis as a tool for teaching electromagnetic field theory.

#### **B.1.1 Development of Differential Forms**

Cartan and others developed the calculus of differential forms in the early 1900's. A differential form is a quantity that can be integrated, including differentials. More precisely, a differential form is a fully covariant, fully antisymmetric tensor. The calculus of differential forms is a self–contained subset of tensor analysis.

Since Cartan's time, the use of forms has spread to many fields of pure and applied mathematics, from differential topology to the theory of differential equations. Differential forms are used by physicists in general relativity [23], quantum field theory [24], thermodynamics [13], mechanics [25], as well as electromagnetics. A section on differential forms is commonplace in mathematical physics texts [26, 27]. Differential forms have been applied to control theory by Hermann [28] and others.

# **B.1.2 Differential Forms in EM Theory**

The laws of electromagnetic field theory as expressed by James Clerk Maxwell in the mid 1800's required dozens of equations. Vector analysis offered a more convenient tool for working with EM theory than earlier methods. Tensor analysis is in turn more concise and general, but is too abstract to give students a conceptual understanding of EM theory. Weyl and Poincaré expressed Maxwell's laws using differential forms early this century. Applied to electromagnetics, differential forms combine much of the generality of tensors with the simplicity and concreteness of vectors.

General treatments of differential forms and EM theory include papers [33], [34], [35], [36], [37], and [41]. Ingarden and Jamiołkowksi [31] is an electrodynamics text using a mix of vectors and differential forms. Parrott [32] employs differential forms to treat advanced electrodynamics. Thirring [12] is a classical field theory text that includes certain applied topics such as waveguides. Bamberg and Sternberg [13] develop a range of topics in mathematical physics, including EM theory via a discussion of discrete forms and circuit theory. Burke [30] treats a range of physics topics using forms, shows how to graphically represent forms, and gives a useful discussion of twisted differential forms. The general relativity text by Misner, Thorne and Wheeler [23] has several chapters on EM theory and differential forms, emphasizing the graphical representation of forms. Flanders [25] treats the calculus of forms and various applications, briefly mentioning electromagnetics.

We note here that many authors, including most of those referenced above, give the spacetime formulation of Maxwell's laws using forms, in which time is included as a differential. We use only the (3+1) representation in this appendix, since the spacetime representation is treated in many references and is not as convenient for various elementary and applied topics. Other formalisms for EM theory are available, including bivectors, quaternions, spinors, and higher Clifford algebras. None of these offer the combination of concrete graphical representations, ease of presentation, and close relationship to traditional vector methods that the calculus of differential forms brings to undergraduate–level electromagnetics.

The tools of applied electromagnetics have begun to be reformulated using differential forms. The authors have developed a convenient representation of electromagnetic boundary conditions [2]. Thirring [12] treats several applications of EM theory using forms, and this dissertation applies differential forms to the Green function theory of complex media.

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# **B.1.3 Pedagogical Advantages of Differential Forms**

As a language for teaching electromagnetics, differential forms offer several important advantages over vector analysis. Vector analysis allows only two types of quantities: scalar fields and vector fields (ignoring inversion properties). In a three–dimensional space, differential forms of four different types are available. This allows flux density and field intensity to have distinct mathematical expressions and graphical representations, providing the student with mental pictures that clearly reveal the different properties of each type of quantity. The physical interpretation of a vector field is often implicitly contained in the choice of operator or integral that acts on it. With differential forms, these properties are directly evident in the type of form used to represent the quantity.

The basic derivative operators of vector analysis are the gradient, curl and divergence. The gradient and divergence lend themselves readily to geometric interpretation, but the curl operation is more difficult to visualize. The gradient, curl and divergence become special cases of a single operator, the exterior derivative and the curl obtains a graphical representation that is as clear as that for the divergence. The physical meanings of the curl operation and the integral expressions of Faraday's and Ampere's laws become so intuitive that the usual order of development can be reversed by introducing Faraday's and Ampere's laws to students first and using these to motivate Gauss's laws.

The Stokes theorem and the divergence theorem have an obvious connection in that they relate integrals over a boundary to integrals over the region inside the boundary, but in the language of vector analysis they appear very different. These theorems are special cases of the generalized Stokes theorem for differential forms, which also has a simple graphical interpretation.

Since 1992, in the Brigham Young University Department of Electrical and Computer Engineering we have incorporated short segments on differential forms into our beginning, intermediate, and graduate electromagnetics courses. In the Fall of 1995, we reworked the entire beginning electromagnetics course, changing emphasis from vector analysis to differential forms. Following the first semester in which the new curriculum was used, students completed a detailed written evaluation. Out of 44 responses, four were partially negative; the rest were in favor of the change to differential forms. Certainly, enthusiasm of students involved in something new increased the likelihood of positive responses, but one fact was clear: pictures of differential forms helped students understand the principles of electromagnetics.

# **B.1.4** Outline

Section B.2 defines differential forms and the degree of a form. Graphical representations for forms of each degree are given, and the differential forms representing the various quantities of electromagnetics are identified. In Sec. B.3 we use these differential forms to express Maxwell's laws in integral form and give graphical interpretations for each of the laws. Section B.4 introduces differential forms in curvilinear coordinate systems. Section B.5 applies Maxwell's laws to find the fields due to sources of basic geometries. In Sec. B.6 we define the exterior derivative, give the generalized Stokes theorem, and express Maxwell's laws in point form. Section B.7 treats boundary conditions using the interior product. Section B.8 provides a summary of the main points made in the appendix.

#### **B.2** Differential Forms and the Electromagnetic Field

In this section we define differential forms of various degrees and identify them with field intensity, flux density, current density, charge density and scalar potential.

A differential form is a quantity that can be integrated, including differentials. 3x dx is a differential form, as are  $x^2y dx dy$  and f(x, y, z) dy dz + g(x, y, z) dz dx. The type of integral called for by a differential form determines its degree. The form 3x dx is integrated under a single integral over a path and so is a 1-form. The form  $x^2y dx dy$  is integrated by a double integral over a surface, so its degree is two. A 3-form is integrated by a triple integral over a volume. 0-forms are functions, "integrated" by evaluation at a point. Table B.1 gives examples of forms of various degrees. The coefficients of the forms can be functions of position, time, and other variables.

**Region of Integration** General Form Degree Example 0-form Point 3x $f(x, y, z, \ldots)$ 1-form Path  $y^2 dx + z dy$  $\alpha_1 dx + \alpha_2 dy + \alpha_3 dz$  $e^{y} dy dz + e^{x} q dz dx$  $\beta_1 dy dz + \beta_2 dz dx + \beta_3 dx dy$ 2-form Surface 3-form Volume (x+y) dx dy dzq dx dy dz

Table B.1: Differential forms of each degree.

# **B.2.1** Representing the Electromagnetic Field with Differential Forms

From Maxwell's laws in integral form, we can readily determine the degrees of the differential forms that will represent the various field quantities. In vector notation,

$$\begin{split} \oint_{P} \mathbf{E} \cdot d\mathbf{l} &= -\frac{d}{dt} \int_{A} \mathbf{B} \cdot d\mathbf{A} \\ \oint_{P} \mathbf{H} \cdot d\mathbf{l} &= \frac{d}{dt} \int_{A} \mathbf{D} \cdot d\mathbf{A} + \int_{A} \mathbf{J} \cdot d\mathbf{A} \\ \oint_{S} \mathbf{D} \cdot d\mathbf{S} &= \int_{V} q dv \\ \oint_{S} \mathbf{B} \cdot d\mathbf{S} &= 0 \end{split}$$

where A is a surface bounded by a path P, V is a volume bounded by a surface S, q is volume charge density, and the other quantities are defined as usual. The electric field intensity is integrated over a path, so that it becomes a 1-form. The magnetic field intensity is also integrated over a path, and becomes a 1-form as well. The electric and magnetic flux densities are integrated over surfaces, and so are 2-forms. The sources are electric current density, which is a 2-form, since it falls under a surface integral, and the volume charge density, which is a 3-form, as it is integrated over a volume. Table B.2 summarizes these forms.

# **B.2.2** 1-Forms; Field Intensity

The usual physical motivation for electric field intensity is the force experienced by a small test charge placed in the field. This leads naturally to the vector representation of the electric field, which might be called the "force picture." Another physical viewpoint for the electric field is the change in potential experienced by a charge as it moves through

Quantity	Form	Degree	Units	Vector/Scalar
Electric Field Intensity	E	1-form	V	Ε
Magnetic Field Intensity	H	1-form	А	Η
Electric Flux Density	D	2-form	С	D
Magnetic Flux Density	B	2-form	Wb	В
Electric Current Density	J	2-form	А	J
Electric Charge Density	$\rho$	3-form	С	q

Table B.2: The differential forms that represent fields and sources.

the field. This leads naturally to the equipotential representation of the field, or the "energy picture." The energy picture shifts emphasis from the local concept of force experienced by a test charge to the global behavior of the field as manifested by change in energy of a test charge as it moves along a path.

Differential forms lead to the "energy picture" of field intensity. A 1-form is represented graphically as surfaces in space [23, 30]. For a conservative field, the surfaces of the associated 1-form are equipotentials. The differential dx produces surfaces perpendicular to the x-axis, as shown in Fig. B.1a. Likewise, dy has surfaces perpendicular to the y-axis and the surfaces of dz are perpendicular to the z axis. A linear combination of these differentials has surfaces that are skew to the coordinate axes. The coefficients of a 1-form determine the spacing of the surfaces per unit length; the greater the magnitude of the coefficients, the more closely spaced are the surfaces. The 1-form 2 dz, shown in Fig. B.1b, has surfaces spaced twice as closely as those of dx in Fig. B.1a.

The surfaces of more general 1-forms can curve, end, or meet each other, depending on the behavior of the coefficients of the form. If surfaces of a 1-form do not meet or end, the field represented by the form is conservative. The field corresponding to the 1-form in Fig. B.1a is conservative; the field in Fig. B.1c is nonconservative.

Just as a line representing the magnitude of a vector has two possible orientations, the surfaces of a 1-form are oriented as well. This is done by specifying one of the two normal directions to the surfaces of the form. The surfaces of 3 dx are oriented in the



Figure B.1: (a) The 1-form dx, with surfaces perpendicular to the x axis and infinite in the y and z directions. (b) The 1-form 2 dz, with surfaces perpendicular to the z-axis and spaced two per unit distance in the zdirection. (c) A more general 1-form, with curved surfaces and surfaces that end or meet each other.

+x direction, and those of -3 dx in the -x direction. The orientation of a form is usually clear from context and is omitted from figures.

Differential forms are by definition the quantities that can be integrated, so it is natural that the surfaces of a 1-form are a graphical representation of path integration. The integral of a 1-form along a path is the number of surfaces pierced by the path (Fig. B.2), taking into account the relative orientations of the surfaces and the path. This simple picture of path integration will provide in the next section a means for visualizing Ampere's and Faraday's laws.

The 1-form  $E_1 dx + E_2 dy + E_3 dz$  is said to be *dual* to the vector field  $E_1 \hat{\mathbf{x}} + E_2 \hat{\mathbf{y}} + E_3 \hat{\mathbf{z}}$ . The field intensity 1-forms E and H are dual to the vectors  $\mathbf{E}$  and  $\mathbf{H}$ .



Figure B.2: A path piercing four surfaces of a 1-form. The integral of the 1-form over the path is four.

Following Deschamps, we take the units of the electric and magnetic field intensity 1-forms to be Volts and Amps, as shown in Table B.2. The differentials are considered to have units of length. Other field and source quantities are assigned units according to this same convention. A disadvantage of Deschamps' system is that it implies in a sense that the metric of space carries units. Alternative conventions are available; Bamberg and Sternberg [13] and others take the units of the electric and magnetic field intensity 1-forms to be V/m and A/m, the same as their vector counterparts, so that the differentials carry no units and the integration process itself is considered to provide a factor of length. If this convention is chosen, the basis differentials of curvilinear coordinate systems (see Sec. B.4) must also be taken to carry no units. This leads to confusion for students, since these basis differentials can include factors of distance. The advantages of this alternative convention are that it is more consistent with the mathematical point of view, in which basis vectors and forms are abstract objects not associated with a particular system of units, and that a field quantity has the same units whether represented by a vector or a differential form. Furthermore, a general differential form may include differentials of functions that do not represent position and so cannot be assigned units of length. The possibility of confusion when using curvilinear coordinates seems to outweigh these considerations, and so we have chosen Deschamps' convention.

With this convention, the electric field intensity 1-form can be taken to have units of energy per charge, or J/C. This supports the "energy picture," in which the electric field represents the change in energy experienced by a charge as it moves through the field. One might argue that this motivation of field intensity is less intuitive than the concept of force experienced by a test charge at a point. While this may be true, the graphical representations of Ampere's and Faraday's laws that will be outlined in Sec. B.3 favor the differential form point of view. Furthermore, the simple correspondence between vectors and forms allows both to be introduced with little additional effort, providing students a more solid understanding of field intensity than they could obtain from one representation alone.

# **B.2.3 2-Forms; Flux Density and Current Density**

Flux density or flow of current can be thought of as tubes that connect sources of flux or current. This is the natural graphical representation of a 2-form, which is drawn as sets of surfaces that intersect to form tubes. The differential dx dy is represented by the surfaces of dx and dy superimposed. The surfaces of dx perpendicular to the x-axis and those of dy perpendicular to the y-axis intersect to produce tubes in the z direction, as illustrated by Fig. B.3a. (To be precise, the tubes of a 2-form have no definite shape: tubes of dxdy have the same density those of [.5 dx][2 dy].) The coefficients of a 2-form give the spacing of the tubes. The greater the coefficients, the more dense the tubes. An arbitrary 2-form has tubes that may curve or converge at a point.

The direction of flow or flux along the tubes of a 2-form is given by the righthand rule applied to the orientations of the surfaces making up the walls of a tube. The orientation of dx is in the +x direction, and dy in the +y direction, so the flux due to dx dy is in the +z direction.

As with 1-forms, the graphical representation of a 2-form is fundamentally related to the integration process. The integral of a 2-form over a surface is the number of tubes passing through the surface, where each tube is weighted positively if its orientation is in the direction of the surface's oriention, and negatively if opposite. This is illustrated in Fig. B.3b.



Figure B.3: (a) The 2-form dx dy, with tubes in the z direction. (b) Four tubes of a 2-form pass through a surface, so that the integral of the 2-form over the surface is four.

As with 1-forms, 2-forms correspond to vector fields in a simple way. An arbitrary 2-form  $D_1 dy dz + D_2 dz dx + D_3 dx dy$  is dual to the vector field  $D_1 \hat{\mathbf{x}} + D_2 \hat{\mathbf{y}} + D_3 \hat{\mathbf{z}}$ , so that the flux density 2-forms D and B are dual to the usual flux density vectors  $\mathbf{D}$  and  $\mathbf{B}$ .

# **B.2.4 3-Forms; Charge Density**

Some scalar physical quantities are densities, and can be integrated over a volume. For other scalar quantities, such as electric potential, a volume integral makes no sense. The calculus of forms distinguishes between these two types of quantities by representing densities as 3-forms. Volume charge density, for example, becomes

$$\rho = q \, dx \, dy \, dz \tag{B.1}$$

where q is the usual scalar charge density in the notation of [33].

A 3-form is represented by three sets of surfaces in space that intersect to form boxes. The density of the boxes is proportional to the coefficient of the 3-form; the greater



Figure B.4: The 3-form dx dy dz, with cubes of side equal to one. The cubes fill all space.

the coefficient, the smaller and more closely spaced are the boxes. A point charge is represented by an infinitesimal box at the location of the charge. The 3-form dx dy dz is the union of three families of planes perpendicular to each of the x, y and z axes. The planes along each of the axes are spaced one unit apart, forming cubes of unit side distributed evenly throughout space, as in Fig. B.4. The orientation of a 3-form is given by specifying the sign of its boxes. As with other differential forms, the orientation is usually clear from context and is omitted from figures.

# **B.2.5** 0-forms; Scalar Potential

0-forms are functions. The scalar potential  $\phi$ , for example, is a 0-form. Any scalar physical quantity that is not a volume density is represented by a 0-form.

# **B.2.6** Summary

The use of differential forms helps students to understand electromagnetics by giving them distinct mental pictures that they can associate with the various fields and sources. As vectors, field intensity and flux density are mathematically and graphically indistinguishable as far as the type of physical quantity they represent. As differential forms, the two types of quantities have graphical representations that clearly express the physical meaning of the field. The surfaces of a field intensity 1-form assign potential change to a path. The tubes of a flux density 2-form give the total flux or flow through a surface. Charge density is also distinguished from other types of scalar quantities by its representation as a 3-form.

#### **B.3** Maxwell's Laws in Integral Form

In this section, we discuss Maxwell's laws in integral form in light of the graphical representations given in the previous section. Using the differential forms defined in Table B.2, Maxwell's laws can be written

$$\oint_{P} E = -\frac{d}{dt} \int_{A} B$$

$$\oint_{P} H = \frac{d}{dt} \int_{A} D + \int_{A} J$$

$$\oint_{S} D = \int_{V} \rho$$

$$\oint_{S} B = 0.$$
(B.2)

The first pair of laws is often more difficult for students to grasp than the second, because the vector picture of curl is not as intuitive as that for divergence. With differential forms, Ampere's and Faraday's laws are graphically very similar to Gauss's laws for the electric and magnetic fields. The close relationship between the two sets of laws becomes clearer.

# **B.3.1** Ampere's and Faraday's Laws

Faraday's and Ampere's laws equate the number of surfaces of a 1-form pierced by a closed path to the number of tubes of a 2-form passing through the path. Each tube of J, for example, must have a surface of H extending away from it, so that any path around the tube pierces the surface of H. Thus, Ampere's law states that tubes of displacement current and electric current are sources for surfaces of H. This is illustrated in Fig. B.5a. Likewise, tubes of time–varying magnetic flux density are sources for surfaces of E.

The illustration of Ampere's law in Fig. B.5a is arguably the most important pedagogical advantage of the calculus of differential forms over vector analysis. Ampere's and Faraday's laws are usually considered the more difficult pair of Maxwell's laws, because vector analysis provides no simple picture that makes the physical meaning of these

laws intuitive. Compare Fig. B.5a to the vector representation of the same field in Fig. B.5b. The vector field appears to "curl" everywhere in space. Students must be convinced that indeed the field has no curl except at the location of the current, using some pedagogical device such as an imaginary paddle wheel in a rotating fluid. The surfaces of H, on the other hand, end only along the tubes of current; where they do not end, the field has no curl. This is the fundamental concept underlying Ampere's and Faraday's laws: tubes of time varying flux or current produce field intensity surfaces.



Figure B.5: (a) A graphical representation of Ampere's law: tubes of current produce surfaces of magnetic field intensity. Any loop around the three tubes of J must pierce three surfaces of H. (b) A cross section of the same magnetic field using vectors. The vector field appears to "curl" everywhere, even though the field has nonzero curl only at the location of the current.

# **B.3.2** Gauss's Laws

Gauss's law for the electric field states that the number of tubes of D flowing out through a closed surface must be equal to the number of boxes of  $\rho$  inside the surface. The boxes of  $\rho$  are sources for the tubes of D, as shown in Fig. B.6. Gauss's law for the magnetic flux density states that tubes of the 2-form B can never end—they must either form closed loops or go off to infinity.

Comparing Figs. B.5a and B.6 shows the close relationship between the two sets of Maxwell's laws. In the same way that flux density tubes are produced by boxes of



Figure B.6: A graphical representation of Gauss's law for the electric flux density: boxes of  $\rho$  produce tubes of D.

electric charge, field intensity surfaces are produced by tubes of the sources on the righthand sides of Faraday's and Ampere's laws. Conceptually, the laws only differ in the degrees of the forms involved and the dimensions of their pictures.

# **B.3.3** Constitutive Relations and the Star Operator

The vector expressions of the constitutive relations for an isotropic medium,

$$\mathbf{D} = \epsilon \mathbf{E}$$
$$\mathbf{B} = \mu \mathbf{H},$$

involve scalar multiplication. With differential forms, we cannot use these same relationships, because D and B are 2-forms, while E and H are 1-forms. An operator that relates forms of different degrees must be introduced.

The Hodge star operator [13, 12] naturally fills this role. As vector spaces, the spaces of 0-forms and 3-forms are both one-dimensional, and the spaces of 1-forms and 2-forms are both three-dimensional. The star operator  $\star$  is a set of isomorphisms between these pairs of vector spaces.

For 1-forms and 2-forms, the star operator satisfies

$$\star dx = dy dz$$
  
$$\star dy = dz dx$$
  
$$\star dz = dx dy.$$

0-forms and 3-forms are related by

$$\star 1 = dx \, dy \, dz.$$

In  $\mathbb{R}^3$ , the star operator is its own inverse, so that  $\star\star\alpha = \alpha$ . A 1-form  $\omega$  is dual to the same vector as the 2-form  $\star\omega$ .

Graphically, the star operator replaces the surfaces of a form with orthogonal surfaces, as in Fig. B.7. The 1-form 3 dx, for example, has planes perpendicular to the *x*-axis. It becomes 3 dy dz under the star operation. This 2-form has planes perpendicular to the *y* and the *z* axes.



# Figure B.7: The star operator relates 1-form surfaces to perpendicular 2-form tubes.

By using the star operator, the constitutive relations can be written as

$$D = \epsilon \star E \tag{B.3}$$

$$B = \mu \star H \tag{B.4}$$

where  $\epsilon$  and  $\mu$  are the permittivity and permeability of the medium. The surfaces of E are perpendicular to the tubes of D, and the surfaces of H are perpendicular to the tubes of B. The following example illustrates the use of these relations.

**Example 1.** Finding *D* due to an electric field intensity.

Let  $E = (dx + dy)e^{ik(x-y)}$  V be the electric field in free space. We wish to find the flux density due to this field. Using the constitutive relationship between D and E,

$$D = \epsilon_0 \star (dx + dy) e^{ik(x-y)}$$
  
=  $\epsilon_0 e^{ik(x-y)} (\star dx + \star dy)$   
=  $\epsilon_0 e^{ik(x-y)} (dy dz + dz dx)$  C.

While we restrict our attention to isotropic media in this appendix, the star operator applies equally well to anisotropic media. As discussed in Ref. [13] and elsewhere, the star operator depends on a metric. If the metric is related to the permittivity or the permeability tensor in an appropriate manner, anisotropic star operators are obtained, and the constitutive relations become  $D = \star_e E$  and  $B = \star_h H$ . Graphically, an anisotropic star operator acts on 1-form surfaces to produce 2-form tubes that intersect the surfaces obliquely rather than orthogonally.

#### **B.3.4** The Exterior Product and the Poynting 2-form

Between the differentials of 2-forms and 3-forms is an implied exterior product, denoted by a wedge  $\wedge$ . The wedge is nearly always omitted from the differentials of a form, especially when the form appears under an integral sign. The exterior product of 1-forms is anticommutative, so that  $dx \wedge dy = -dy \wedge dx$ . As a consequence, the exterior product is in general supercommutative:

$$\alpha \wedge \beta = (-1)^{ab} \beta \wedge \alpha \tag{B.5}$$

where a and b are the degrees of  $\alpha$  and  $\beta$ , respectively. One usually converts the differentials of a form to right–cyclic order using (B.5).

As a consequence of (B.5), any differential form with a repeated differential vanishes. In a three-dimensional space each term of a *p*-form will always contain a repeated differential if p > 3, so there are no nonzero *p*-forms for p > 3.

The exterior product of two 1-forms is analogous to the vector cross product. With vector analysis, it is not obvious that the cross product of vectors is a different type of quantity than the factors. Under coordinate inversion,  $\mathbf{a} \times \mathbf{b}$  changes sign relative to a vector with the same components, so that  $\mathbf{a} \times \mathbf{b}$  is a pseudovector. With forms, the distinction between  $a \wedge b$  and a or b individually is clear.

The exterior product of a 1-form and a 2-form corresponds to the dot product. The coefficient of the resulting 3-form is equal to the dot product of the vector fields dual to the 1-form and 2-form in the euclidean metric.

Combinations of cross and dot products are somewhat difficult to manipulate algebraically, often requiring the use of tabulated identities. Using the supercommutativity of the exterior product, the student can easily manipulate arbitrary products of forms. For example, the identities

$$\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C}) = \mathbf{C} \cdot (\mathbf{A} \times \mathbf{B}) = \mathbf{B} \cdot (\mathbf{C} \times \mathbf{A})$$

are in the euclidean metric equivalent to relationships which are easily obtained from (B.5). Factors in any exterior product can be interchanged arbitrarily as long as the sign of the product is changed according to (B.5).

Consider the exterior product of the 1-forms E and H,

$$\begin{split} E \wedge H &= (E_1 \, dx + E_2 \, dy + E_3 \, dz) \wedge (H_1 \, dx + H_2 \, dy + H_3 \, dz) \\ &= E_1 H_1 \, dx \, dx + E_1 H_2 \, dx \, dy + E_1 H_3 \, dx \, dz \\ &\quad + E_2 H_1 \, dy \, dx + E_2 H_2 \, dy \, dy + E_2 H_3 \, dy \, dz \\ &\quad + E_3 H_1 \, dz \, dx + E_3 H_2 \, dz \, dy + E_3 H_3 \, dz \, dz \\ &= (E_2 H_3 - E_3 H_2) \, dy \, dz + (E_3 H_1 - E_1 H_3) \, dz \, dx + (E_1 H_2 - E_2 H_1) \, dx \, dy. \end{split}$$

This is the Poynting 2-form S. For complex fields,  $S = E \wedge H^*$ . For time-varying fields, the tubes of this 2-form represent flow of electromagnetic power, as shown in Fig. B.8. The sides of the tubes are the surfaces of E and H. This gives a clear geometrical interpretation to the fact that the direction of power flow is orthogonal to the orientations of both E and H.



Figure B.8: The Poynting power flow 2-form  $S = E \wedge H$ . Surfaces of the 1-forms E and H are the sides of the tubes of S.

**Example 2.** The Poynting 2-form due to a plane wave.

Consider a plane wave propagating in free space in the z direction, with the time-harmonic electric field  $E = E_0 dx$  V in the x direction. The Poynting 2-form is

$$S = E \wedge H$$
  
=  $E_0 dx \wedge \frac{E_0}{\eta_0} dy$   
=  $\frac{E_0^2}{\eta_0} dx dy$  W

where  $\eta_0$  is the wave impedance of free space.

# **B.3.5** Energy Density

The exterior products  $E \wedge D$  and  $H \wedge B$  are 3-forms that represent the density of electromagnetic energy. The energy density 3-form w is defined to be

$$w = \frac{1}{2} \left( E \wedge D + H \wedge B \right) \tag{B.6}$$

The volume integral of w gives the total energy stored in a region of space by the fields present in the region.

Fig. B.9 shows the energy density 3-form between the plates of a capacitor, where the upper and lower plates are equally and oppositely charged. The boxes of 2w are the intersection of the surfaces of E, which are parallel to the plates, with the tubes of D, which extend vertically from one plate to the other.



Figure B.9: The 3-form 2w due to fields inside a parallel plate capacitor with oppositely charged plates. The surfaces of E are parallel to the top and bottom plates. The tubes of D extend vertically from charges on one plate to opposite charges on the other. The tubes and surfaces intersect to form cubes of  $2\omega$ , one of which is outlined in the figure.

# **B.4** Curvilinear Coordinate Systems

In this section, we give the basis differentials, the star operator, and the correspondence between vectors and forms for cylindrical, spherical, and generalized orthogonal coordinates.

## **B.4.1** Cylindrical Coordinates

The differentials of the cylindrical coordinate system are  $d\rho$ ,  $\rho d\phi$  and dz. Each of the basis differentials is considered to have units of length. The general 1-form

$$A\,d\rho + B\rho\,d\phi + C\,dz\tag{B.7}$$

is dual to the vector

$$A\hat{\rho} + B\hat{\phi} + C\hat{\mathbf{z}}.\tag{B.8}$$

The general 2-form

$$A\rho \, d\phi \wedge \, dz + B \, dz \wedge \, d\rho + C \, d\rho \wedge \rho \, d\phi \tag{B.9}$$

is dual to the same vector. The 2-form  $d\rho \ d\phi$ , for example, is dual to the vector  $(1/\rho)\hat{\mathbf{z}}$ .

Differentials must be converted to basis elements before the star operator is applied. The star operator in cylindrical coordinates acts as follows:

$$\star d\rho = \rho \, d\phi \wedge dz$$
$$\star \rho \, d\phi = dz \wedge d\rho$$
$$\star dz = d\rho \wedge \rho \, d\phi.$$

Also,  $\star 1 = \rho \, d\rho \, d\phi \, dz$ . As with the rectangular coordinate system,  $\star \star = 1$ . The star operator applied to  $d\phi \, dz$ , for example, yields  $(1/\rho) \, d\rho$ .

Fig. B.10 shows the pictures of the differentials of the cylindrical coordinate system. The 2-forms can be obtained by superimposing these surfaces. Tubes of  $dz \wedge d\rho$ , for example, are square rings formed by the union of Figs. B.10a and B.10c.



Figure B.10: Surfaces of (a)  $d\rho$ , (b)  $d\phi$  scaled by  $3/\pi$ , and (c) dz.

# **B.4.2** Spherical Coordinates

The basis differentials of the spherical coordinate system are in right-cyclic order, dr,  $r d\theta$  and  $r \sin \theta d\phi$ , each having units of length. The 1-form

$$A\,dr + Br\,d\theta + Cr\sin\theta\,d\phi\tag{B.10}$$

and the 2-form

$$Ar \, d\theta \wedge r \sin \theta \, d\phi + Br \sin \theta \, d\phi \wedge \, dr + C \, dr \wedge r \, d\theta \tag{B.11}$$

are both dual to the vector

$$A\hat{\mathbf{r}} + B\hat{\theta} + C\hat{\phi} \tag{B.12}$$

so that  $d\theta \ d\phi$ , for example, is dual to the vector  $\hat{\mathbf{r}}/(r^2 \sin \theta)$ .

As in the cylindrical coordinate system, differentials must be converted to basis elements before the star operator is applied. The star operator acts on 1-forms and 2-forms as follows:

$$\star dr = r \, d\theta \wedge r \sin \theta \, d\phi$$
$$\star r \, d\theta = r \sin \theta \, d\phi \wedge dr$$
$$\star r \sin \theta \, d\phi = dr \wedge r \, d\theta$$

Again,  $\star \star = 1$ . The star operator applied to one is  $\star 1 = r^2 \sin \theta \, dr \, d\theta \, d\phi$ . Fig. B.11 shows the pictures of the differentials of the spherical coordinate system; pictures of 2-forms can be obtained by superimposing these surfaces.



Figure B.11: Surfaces of (a) dr, (b)  $d\theta$  scaled by  $10/\pi$ , and (c)  $d\phi$  scaled by  $3/\pi$ .

## **B.4.3** Generalized Orthogonal Coordinates

Let the location of a point be given by (u, v, w) such that the tangents to each of the coordinates are mutually orthogonal. Define a function  $h_1$  such that the integral of  $h_1 du$  along any path with v and w constant gives the length of the path. Define  $h_2$  and  $h_3$  similarly. Then the basis differentials are

$$h_1 du, h_2 dv, h_3 dw.$$
 (B.13)

The 1-form  $Ah_1 du + Bh_2 dv + Ch_3 dw$  and the 2-form  $Ah_2h_3 dv \wedge dw + Bh_3h_1 dw \wedge du + Ch_1h_2 du \wedge dv$  are both dual to the vector  $A\hat{\mathbf{u}} + B\hat{\mathbf{v}} + C\hat{\mathbf{w}}$ . The star operator on 1-forms and 2-forms satisfies

$$\star (Ah_1 \, du + Bh_2 \, dv + Ch_3 \, dw) = Ah_2h_3 \, dv \wedge \, dw + Bh_3h_1 \, dw \wedge \, du + Ch_1h_2 \, du \wedge \, dv$$
(B.14)

For 0-forms and 3-forms,  $\star 1 = h_1 h_2 h_3 du dv dw$ .

# **B.5** Electrostatics and Magnetostatics

In this section we treat several of the usual elementary applications of Maxwell's laws in integral form. We find the electric flux due to a point charge and a line charge using Gauss's law for the electric field. Ampere's law is used to find the magnetic fields produced by a line current.

#### **B.5.1** Point Charge

By symmetry, the tubes of flux from a point charge Q must extend out radially from the charge (Fig. B.12), so that

$$D = D_0 r^2 \sin \theta \, d\theta \, d\phi \tag{B.15}$$

To apply Gauss law  $\oint_S D = \int_V \rho$ , we choose S to be a sphere enclosing the charge. The right-hand side of Gauss's law is equal to Q, and the left-hand side is

$$\oint_S D = \int_0^{2\pi} \int_0^{\pi} D_0 r^2 \sin \theta \, d\theta \, d\phi$$
$$= 4\pi r^2 D_0.$$

Solving for  $D_0$  and substituting into (B.15),

$$D = \frac{Q}{4\pi r^2} r \, d\theta \, r \sin \theta \, d\phi \, \mathbf{C} \tag{B.16}$$

for the electric flux density due to the point charge. This can also be written

$$D = \frac{Q}{4\pi} \sin \theta \ d\theta \ d\phi \ \mathbf{C}. \tag{B.17}$$

Since  $4\pi$  is the total amount of solid angle for a sphere and  $\sin \theta \, d\theta \, d\phi$  is the differential element of solid angle, this expression matches Fig. B.12 in showing that the amount of flux per solid angle is constant.



Figure B.12: Electric flux density due to a point charge. Tubes of D extend away from the charge.

# **B.5.2** Line Charge

For a line charge with charge density  $\rho_l$  C/m, by symmetry tubes of flux extend out radially from the line, as shown in Fig. B.13. The tubes are bounded by the surfaces of  $d\phi$  and dz, so that D has the form

$$D = D_0 \, d\phi \, dz. \tag{B.18}$$

Let S be a cylinder of height b with the line charge along its axis. The right-hand side of Gauss's law is

$$\int_{V} \rho = \int_{0}^{b} \rho_{l} dz$$
$$= b \rho_{l}.$$

The left-hand side is

$$\oint_S D = \int_0^b \int_0^{2\pi} D_0 \, d\phi \, dz$$
$$= 2\pi b D_0.$$

Solving for  $D_0$  and substituting into (B.18), we obtain

$$D = \frac{\rho_l}{2\pi} \, d\phi \, dz \quad \mathbf{C} \tag{B.19}$$

for the electric flux density due to the line charge.



Figure B.13: Electric flux density due to a line charge. Tubes of D extend radially away from the vertical line of charge.

# **B.5.3** Line Current

If a current  $I_l$  A flows along the z-axis, sheets of the H 1-form will extend out radially from the current, as shown in Fig. B.14. These are the surfaces of  $d\phi$ , so that by symmetry,

$$H = H_0 \, d\phi \tag{B.20}$$

where  $H_0$  is a constant we need to find using Ampere's law. We choose the path P in Ampere's law  $\oint_P H = \frac{d}{dt} \int_A D + \int_A J$  to be a loop around the z-axis. Assuming that D = 0, the right-hand side of Ampere's law is equal to  $I_l$ . The left-hand side is the integral of H over the loop,

$$\oint_P H = \int_0^{2\pi} H_0 \, d\phi$$
$$= 2\pi H_0.$$

The magnetic field intensity is then

$$H = \frac{I_l}{2\pi} d\phi \quad \mathbf{A} \tag{B.21}$$

for the line current source.



Figure B.14: Magnetic field intensity H due to a line current.

# **B.6** The Exterior Derivative and Maxwell's Laws in Point Form

In this section we introduce the exterior derivative and the generalized Stokes theorem and use these to express Maxwell's laws in point form. The exterior derivative is a single operator which has the gradient, curl, and divergence as special cases, depending on the degree of the differential form on which the exterior derivative acts. The exterior derivative has the symbol d, and can be written formally as

$$d \equiv \frac{\partial}{\partial x} dx + \frac{\partial}{\partial y} dy + \frac{\partial}{\partial z} dz.$$
 (B.22)

The exterior derivative can be thought of as implicit differentiation with new differentials introduced from the left.

# **B.6.1** Exterior Derivative of 0-forms

Consider the 0-form f(x, y, z). If we implicitly differentiate f with respect to each of the coordinates, we obtain

$$df = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy + \frac{\partial f}{\partial z} dz.$$
 (B.23)

which is a 1-form, the exterior derivative of f. Note that the differentials dx, dy, and dz are the exterior derivatives of the coordinate functions x, y, and z. The 1-form df is dual to the gradient of f.

If  $\phi$  represents a scalar electric potential, the negative of its exterior derivative is electric field intensity:

$$E = -d\phi.$$

As noted earlier, the surfaces of the 1-form E are equipotentials, or level sets of the function  $\phi$ , so that the exterior derivative of a 0-form has a simple graphical interpretation.

# **B.6.2** Exterior Derivative of 1-forms

The exterior derivative of a 1-form is analogous to the vector curl operation. If E is an arbitrary 1-form  $E_1 dx + E_2 dy + E_3 dz$ , then the exterior derivative of E is

$$dE = \left( \frac{\partial}{\partial x} E_1 \, dx + \frac{\partial}{\partial y} E_1 \, dy + \frac{\partial}{\partial z} E_1 \, dz \right) \, dx \\ + \left( \frac{\partial}{\partial x} E_2 \, dx + \frac{\partial}{\partial y} E_2 \, dy + \frac{\partial}{\partial z} E_2 \, dz \right) \, dy \\ + \left( \frac{\partial}{\partial x} E_3 \, dx + \frac{\partial}{\partial y} E_3 \, dy + \frac{\partial}{\partial z} E_3 \, dz \right) \, dz$$

Using the antisymmetry of the exterior product, this becomes

$$dE = \left(\frac{\partial E_3}{\partial y} - \frac{\partial E_2}{\partial z}\right) dy \, dz + \left(\frac{\partial E_1}{\partial z} - \frac{\partial E_3}{\partial x}\right) dz \, dx + \left(\frac{\partial E_2}{\partial x} - \frac{\partial E_1}{\partial y}\right) dx \, dy, \qquad (B.24)$$

which is a 2-form dual to the curl of the vector field  $E_1 \hat{\mathbf{x}} + E_2 \hat{\mathbf{y}} + E_3 \hat{\mathbf{z}}$ .

Any 1-form E for which dE = 0 is called *closed* and represents a conservative field. Surfaces representing different potential values can never meet. If  $dE \neq 0$ , the field is non-conservative, and surfaces meet or end wherever the exterior derivative is nonzero.

# **B.6.3** Exterior Derivative of 2-forms

The exterior derivative of a 2-form is computed by the same rule as for 0-forms and 1-forms: take partial derivatives by each coordinate variable and add the corresponding differential on the left. For an arbitrary 2-form B,

$$dB = d(B_1 dy dz + B_2 dz dx + B_3 dx dy)$$
  
=  $\left(\frac{\partial}{\partial x}B_1 dx + \frac{\partial}{\partial y}B_1 dy + \frac{\partial}{\partial z}B_1 dz\right) dy dz$   
+  $\left(\frac{\partial}{\partial x}B_2 dx + \frac{\partial}{\partial y}B_2 dy + \frac{\partial}{\partial z}B_2 dz\right) dz dx$   
+  $\left(\frac{\partial}{\partial x}B_3 dx + \frac{\partial}{\partial y}B_3 dy + \frac{\partial}{\partial z}B_3 dz\right) dx dy$   
=  $\left(\frac{\partial B_1}{\partial x} + \frac{\partial B_2}{\partial y} + \frac{\partial B_3}{\partial z}\right) dx dy dz$ 

where six of the terms vanish due to repeated differentials. The coefficient of the resulting 3-form is the divergence of the vector field dual to B.

#### **B.6.4** Properties of the Exterior Derivative

Because the exterior derivative unifies the gradient, curl, and divergence operators, many common vector identities become special cases of simple properties of the exterior derivative. The equality of mixed partial derivatives leads to the identity

$$dd = 0, \tag{B.25}$$

so that the exterior derivative applied twice yields zero. This relationship is equivalent to the vector relationships  $\nabla \times (\nabla f) = 0$  and  $\nabla \cdot (\nabla \times \mathbf{A}) = 0$ . The exterior derivative also obeys the product rule

$$d(\alpha \wedge \beta) = d\alpha \wedge \beta + (-1)^p \alpha \wedge d\beta \tag{B.26}$$

where p is the degree of  $\alpha$ . A special case of (B.26) is

$$\nabla \cdot (\mathbf{A} \times \mathbf{B}) = \mathbf{B} \cdot (\nabla \times \mathbf{A}) - \mathbf{A} \cdot (\nabla \times \mathbf{B}).$$

These and other vector identities are often placed in reference tables; by contrast, (B.25) and (B.26) are easily remembered.

The exterior derivative in cylindrical coordinates is

$$d = \frac{\partial}{\partial \rho} d\rho + \frac{\partial}{\partial \phi} d\phi + \frac{\partial}{\partial z} dz$$
 (B.27)

which is the same as for rectangular coordinates but with the coordinates  $\rho$ ,  $\phi$ , z in the place of x, y, z. Note that the exterior derivative does not require the factor of  $\rho$  that is involved in converting forms to vectors and applying the star operator. In spherical coordinates,

$$d = \frac{\partial}{\partial r} dr + \frac{\partial}{\partial \theta} d\theta + \frac{\partial}{\partial \phi} d\phi$$
 (B.28)

where the factors r and  $r \sin \theta$  are not found in the exterior derivative operator. The exterior derivative is

$$d = \frac{\partial}{\partial u} du + \frac{\partial}{\partial v} dv + \frac{\partial}{\partial w} dw$$
 (B.29)

in general orthogonal coordinates. The exterior derivative is much easier to apply in curvilinear coordinates than the vector derivatives; there is no need for reference tables of derivative formulas in various coordinate systems.

# **B.6.5** The Generalized Stokes Theorem

The exterior derivative satisfies the generalized Stokes theorem, which states that for any *p*-form  $\omega$ ,

$$\int_{M} d\omega = \oint_{bd\,M} \omega \tag{B.30}$$

where M is a (p + 1)-dimensional region of space and bd M is its boundary. If  $\omega$  is a 0-form, then the Stokes theorem becomes  $\int_a^b df = f(b) - f(a)$ . This is the fundamental theorem of calculus.

If  $\omega$  is a 1-form, then bd M is a closed loop and M is a surface that has the path as its boundary. This case is analogous to the vector Stokes theorem. Graphically, the number of surfaces of  $\omega$  pierced by the loop equals the number of tubes of the 2-form  $d\omega$  that pass through the loop (Fig. B.15).

If  $\omega$  is a 2-form, then bd M is a closed surface and M is the volume inside it. The Stokes theorem requires that the number of tubes of  $\omega$  that cross the surface equal the



Figure B.15: The Stokes theorem for  $\omega$  a 1-form. (a) The loop bdM pierces three of the surfaces of  $\omega$ . (b) Three tubes of  $d\omega$  pass through any surface M bounded by the loop bdM.

number of boxes of  $d\omega$  inside the surface, as shown in Fig. B.16. This is equivalent to the vector divergence theorem.

Compared to the usual formulations of these theorems,

$$f(b) - f(a) = \int_{a}^{b} \frac{\partial f}{\partial x} dx$$
  
$$\oint_{bdA} \mathbf{E} \cdot d\mathbf{l} = \int_{A} \nabla \times \mathbf{E} \cdot d\mathbf{A}$$
  
$$\oint_{bdV} \mathbf{D} \cdot d\mathbf{S} = \int_{V} \nabla \cdot \mathbf{D} dv$$

the generalized Stokes theorem is simpler in form and hence easier to remember. It also makes clear that the vector Stokes theorem and the divergence theorem are higher-dimensional statements of the fundamental theorem of calculus.



Figure B.16: Stokes theorem for  $\omega$  a 2-form. (a) Four tubes of the 2-form  $\omega$  pass through a surface. (b) The same number of boxes of the 3-form  $d\omega$  lie inside the surface.

# **B.6.6 Faraday's and Ampere's Laws in Point Form**

Faraday's law in integral form is

$$\oint_{P} E = -\frac{d}{dt} \int_{A} B.$$
(B.31)

Using the Stokes theorem, taking M to be the surface A, we can relate the path integral of E to the surface integral of the exterior derivative of E,

$$\oint_{P} E = \int_{A} dE. \tag{B.32}$$

By Faraday's law,

$$\int_{A} dE = -\frac{d}{dt} \int_{A} B.$$
 (B.33)

For sufficiently regular forms E and B, we have that

$$dE = -\frac{\partial B}{\partial t} \tag{B.34}$$

since (B.33) is valid for all surfaces A. This is Faraday's law in point form. This law states that new surfaces of E are produced by tubes of time–varying magnetic flux.

Using the same argument, Ampere's law becomes

$$dH = \frac{\partial D}{\partial t} + J. \tag{B.35}$$

Ampere's law shows that new surfaces of H are produced by tubes of time–varying electric flux or electric current.

# **B.6.7** Gauss's Laws in Point Form

Gauss's law for the electric flux density is

$$\oint_{S} D = \int_{V} \rho. \tag{B.36}$$

The Stokes theorem with M as the volume V and bd M as the surface S shows that

$$\oint_{S} D = \int_{V} dD. \tag{B.37}$$

Using Gauss's law in integral form (B.36),

$$\int_{V} dD = \int_{V} \rho. \tag{B.38}$$

We can then write

$$dD = \rho. \tag{B.39}$$

This is Gauss's law for the electric field in point form. Graphically, this law shows that tubes of electric flux density can end only on electric charges. Similarly, Gauss's law for the magnetic field is

$$dB = 0. \tag{B.40}$$

This law requires that tubes of magnetic flux density never end; they must form closed loops or extend to infinity.

#### **B.6.8** Poynting's Theorem

Using Maxwell's laws, we can derive a conservation law for electromagnetic energy. The exterior derivative of S is

$$dS = d(E \wedge H)$$
$$= (dE) \wedge H - E \wedge (dH)$$
Using Ampere's and Faraday's laws, this can be written

$$dS = -\frac{\partial B}{\partial t} \wedge H - E \wedge \frac{\partial D}{\partial t} - E \wedge J$$
(B.41)

Finally, using the definition (B.6) of w, this becomes

$$dS = -\frac{\partial w}{\partial t} - E \wedge J. \tag{B.42}$$

At a point where no sources exist, a change in stored electromagnetic energy must be accompanied by tubes of S that represent flow of energy towards or away from the point.

#### **B.6.9** Integrating Forms by Pullback

We have seen in previous sections that differential forms give integration a clear graphical interpretation. The use of differential forms also results in several simplifications of the integration process itself. Integrals of vector fields require a metric; integrals of differential forms do not. The method of pullback replaces the computation of differential length and surface elements that is required before a vector field can be integrated.

Consider the path integral

$$\int_{P} \mathbf{E} \cdot d\mathbf{l}.$$
 (B.43)

The dot product of  $\mathbf{E}$  with  $d\mathbf{l}$  produces a 1-form with a single differential in the parameter of the path P, allowing the integral to be evaluated. The integral of the 1-form E dual to  $\mathbf{E}$  over the same path is computed by the method of *pullback*, as change of variables for differential forms is commonly termed. Let the path P be parameterized by

$$x = p_1(t), y = p_2(t), z = p_3(t)$$

for a < t < b. The pullback of E to the path P is denoted  $P^*E$ , and is defined to be

$$P^*E = P^*(E_1 dx + E_2 dy + E_3 dz)$$
  
=  $E_1(p_1, p_2, p_3)dp_1 + E_2(p_1, p_2, p_3)dp_2 + E_3(p_1, p_2, p_3)dp_3.$   
=  $\left(E_1(p_1, p_2, p_3)\frac{\partial p_1}{\partial t} + E_2(p_1, p_2, p_3)\frac{\partial p_2}{\partial t} + E_3(p_1, p_2, p_3)\frac{\partial p_3}{\partial t}\right) dt$ 

Using the pullback of E, we convert the integral over P to an integral in t over the interval [a, b],

$$\int_{P} E = \int_{a}^{b} P^* E \tag{B.44}$$

Components of the Jacobian matrix of the coordinate transform from the original coordinate system to the parameterization of the region of integration enter naturally when the exterior derivatives are performed. Pullback works similarly for 2-forms and 3-forms, allowing evaluation of surface and volume integrals by the same method. The following example illustrates the use of pullback.

# Example 3. Work required to move a charge through an electric field.

Let the electric field intensity be given by  $E = 2xy dx + x^2 dy - dz$ . A charge of q = 1 C is transported over the path P given by  $(x = t^2, y = t, z = 1 - t^3)$ from t = 0 to t = 1. The work required is given by

$$W = -q \int_{P} 2xy \, dx + x^2 \, dy - dz \tag{B.45}$$

which by Eq. (B.44) is equal to

$$= -q \int_0^1 P^* (2xy \, dx + x^2 \, dy - dz)$$

where  $P^*E$  is the pullback of the field 1-form to the path P,

$$P^*E = 2(t^2)(t)2t dt + (t^2)^2 dt - (-3t^2) dt$$
$$= (5t^4 + 3t^2) dt.$$

Integrating this new 1-form in t over [0, 1], we obtain

$$W = -\int_0^1 (5t^4 + 3t^2)dt = -2 J$$

as the total work required to move the charge along P.

#### **B.6.10** Existence of Graphical Representations

With the exterior derivative, a condition can be given for the existence of the graphical representations of Sec. B.2. These representations do not correspond to the usual "tangent space" picture of a vector field, but rather are analogous to the integral curves of a vector field. Obtaining the graphical representation of a differential form as a family of surfaces is in general nontrivial, and is closely related to Pfaff's problem [51]. By the solution to Pfaff's problem, each differential form may be represented graphically in two dimensions as families of lines. In three dimensions, a 1-form  $\omega$  can be represented as surfaces if the rotation  $\omega \wedge d\omega$  is zero. If  $\omega \wedge d\omega \neq 0$ , then there exist local coordinates for which  $\omega$  has the form du + v dw, so that it is the sum of two 1-forms, both of which can be graphically represented as surfaces.

An arbitrary, smooth 2-form in  $\mathbb{R}^3$  can be written locally in the form  $fdg \wedge dh$ , so that the 2-form consists of tubes of  $dg \wedge dh$  scaled by f.

# **B.6.11** Summary

Throughout this section, we have noted various aspects of the calculus of differential forms that simplify manipulations and provide insight into the principles of electromagnetics. The exterior derivative behaves differently depending on the degree of the form it operates on, so that physical properties of a field are encoded in the type of form used to represent it, rather than in the type of operator used to take its derivative. The generalized Stokes theorem gives the vector Stokes theorem and the divergence theorem intuitive graphical interpretations that illuminate the relationship between the two theorems. While of lesser pedagogical importance, the algebraic and computational advantages of forms cited in this section also aid students by reducing the need for reference tables or memorization of identities.

### **B.7** The Interior Product and Boundary Conditions

Boundary conditions can be expressed using a combination of the exterior and interior products. The same operator is used to express boundary conditions for field intensities and flux densities, and in both cases the boundary conditions have simple graphical interpretations.

#### **B.7.1** The Interior Product

The interior product has the symbol  $\bot$ . Graphically, the interior product removes the surfaces of the first form from those of the second. The interior product  $dx \bot dy$  is zero, since there are no dx surfaces to remove. The interior product of dx with itself is one. The interior product of dx and dx dy is  $dx \bot dx dy = dy$ . To compute the interior product  $dy \bot dx dy$ , the differential dy must be moved to the left of dx dy before it can be removed, so that

$$dy \,\lrcorner\, dx \, dy = -dy \,\lrcorner\, dy \, dx$$
$$= -dx.$$

The interior product of arbitrary 1-forms can be found by linearity from the relationships

$$dx \sqcup dx = 1, \quad dx \sqcup dy = 0, \quad dx \sqcup dz = 0$$
  

$$dy \sqcup dx = 0, \quad dy \sqcup dy = 1, \quad dy \sqcup dz = 0$$
  

$$dz \sqcup dx = 0, \quad dz \sqcup dy = 0, \quad dz \sqcup dz = 1.$$
  
(B.46)

The interior product of a 1-form and a 2-form can be found using

$$dx \sqcup dy \wedge dz = 0, \quad dx \sqcup dz \wedge dx = -dz, \quad dx \sqcup dx \wedge dy = dy$$
$$dy \sqcup dy \wedge dz = dz, \quad dy \sqcup dz \wedge dx = 0, \quad dy \sqcup dx \wedge dy = -dx \quad (B.47)$$
$$dz \sqcup dy \wedge dz = -dy, \quad dz \sqcup dz \wedge dx = dx, \quad dz \sqcup dx \wedge dy = 0.$$

The following examples illustrate the use of the interior product.

**Example 4.** The Interior Product of two 1-forms

The interior product of  $a = 3x \, dx - y \, dz$  and  $b = 4 \, dy + 5 \, dz$  is

$$a \,\lrcorner b = (3x \, dx - y \, dz) \,\lrcorner (4 \, dy + 5 \, dz)$$
$$= 12x \, dx \,\lrcorner \, dy + 15x \, dx \,\lrcorner \, dz - 4y \, dz \,\lrcorner \, dy - 5y \, dz \,\lrcorner \, dz$$
$$= -5y$$

which is the dot product  $\mathbf{a} \cdot \mathbf{b}$  of the vectors dual to the 1-forms a and b.

**Example 5.** The Interior Product of a 1-form and a 2-form

The interior product of  $a = 3x \, dx - y \, dz$  and  $c = 4 \, dz \, dx + 5 \, dx \, dy$  is

$$a \sqcup c = (3x \, dx - y \, dz) \sqcup (4 \, dz \, dx + 5 \, dx \, dy)$$
  
$$= 12x \, dx \sqcup dz \, dx + 15x \, dx \sqcup dx \, dy - 4y \, dz \sqcup dz \, dx - 5y \, dz \sqcup dx \, dy$$
  
$$= -12x \, dz + 15x \, dy - 4y \, dx$$

which is the 1-form dual to  $-\mathbf{a} \times \mathbf{c}$ , where  $\mathbf{a}$  and  $\mathbf{c}$  are dual to a and c.

The interior product can be related to the exterior product using the star operator. The interior product of arbitrary forms *a* and *b* is

$$a \,\lrcorner b = \star (\star b \land a) \tag{B.48}$$

which can be used to compute the interior product in curvilinear coordinate systems. (This formula shows the metric dependence of the interior product as we have defined it; the interior product is usually defined to be the contraction of a vector with a form, which is independent of any metric.) The interior and exterior products satisfy the identity

$$\alpha = n \wedge (n \, \lrcorner \, \alpha) + n \, \lrcorner \, (n \wedge \alpha) \tag{B.49}$$

where n is a 1-form and  $\alpha$  is arbitrary.

The Lorentz force law can be expressed using the interior product. The force 1-form F is

$$F = q(E - \mathbf{v} \,\lrcorner\, B) \tag{B.50}$$

where  $\mathbf{v}$  is the velocity of a charge q, and the interior product can be computed by finding the 1-form dual to  $\mathbf{v}$  and using the rules given above. F is dual to the usual force vector  $\mathbf{F}$ . The force 1-form has units of energy, and does not have as clear a physical interpretation as the usual force vector. In this case we prefer to work with the vector dual to F, rather than F itself. Force, like displacement and velocity, is naturally a vector quantity.

## **B.7.2 Boundary Conditions**

A boundary can be specified as the set of points satisfying f(x, y, z) = 0 for some suitable function f. The surface normal 1-form is defined to be the normalized exterior derivative of f,

$$n = \frac{df}{\sqrt{(df \,\lrcorner\, df)}}.\tag{B.51}$$

The surfaces of n are parallel to the boundary. Using a subscript 1 to denote the region where f > 0, and a subscript 2 for f < 0, the four electromagnetic boundary conditions can be written [2]

$$n \sqcup (n \land (E_1 - E_2)) = 0$$
  

$$n \sqcup (n \land (H_1 - H_2)) = J_s$$
  

$$n \sqcup (n \land (D_1 - D_2)) = \rho_s$$
  

$$n \sqcup (n \land (B_1 - B_2)) = 0$$

where  $J_s$  is the surface current density 1-form and  $\rho_s$  is the surface charge density 2-form. The form  $n \, \lrcorner \, (n \land \omega)$  is the component of  $\omega$  which has surfaces perpendicular to the boundary and integrates to the same value as  $\omega$  over any region lying in the boundary.

#### **B.7.3** Surface Current

The action of the operator  $n \, \lrcorner n \land$  can be interpreted graphically, leading to a simple picture of the field intensity boundary conditions. Consider the field discontinuity  $H_1 - H_2$  shown in Fig. B.17a. The exterior product of n and  $H_1 - H_2$  is a 2-form with tubes that run parallel to the boundary, as shown in Fig. B.17b. The component of  $H_1 - H_2$  with surfaces parallel to the boundary is removed. The interior product  $n \, \lrcorner (n \land (H_1 - H_2))$ 

removes the surfaces parallel to the boundary, leaving only surfaces perpendicular to the boundary, as in Fig. B.17c. Current flows along the lines where the surfaces intersect the boundary. The direction of flow along the lines of the 1-form can be found using the right-hand rule on the direction of  $H_1 - H_2$  in region 1 above the boundary.



Figure B.17: (a) The 1-form  $H_1-H_2$ . (b) The 2-form  $n \wedge (H_1-H_2)$ . (c) The 1-form  $J_s$ , represented by lines on the boundary. Current flows along the lines.

The field intensity boundary conditions are intuitive: the boundary condition for magnetic field intensity requires that surfaces of the 1-form  $H_1 - H_2$  end along lines of the surface current density 1-form  $J_s$ , as can be seen in Fig. B.17. The surfaces of  $E_1 - E_2$ cannot intersect a boundary at all, so that they must be parallel to the boundary.

Unlike other electromagnetic quantities,  $J_s$  is not dual to the vector  $\mathbf{J}_s$ . The direction of  $\mathbf{J}_s$  is parallel to the lines of  $J_s$  in the boundary, as shown in Fig. B.17c. ( $J_s$  is a twisted differential form, so that under coordinate inversion it transforms with a minus sign relative to a nontwisted 1-form. This property is discussed in detail in Refs. [30, 2, 44]. Operationally, the distinction can be ignored as long as one remains in right–handed coordinates.)  $J_s$  is natural both mathematically and geometrically as a representation of

surface current density. The expression for current through a path using the vector surface current density is

$$I = \int_{P} \mathbf{J}_{s} \cdot (\hat{\mathbf{n}} \times d\hat{\mathbf{l}})$$
(B.52)

where  $\hat{\mathbf{n}}$  is a surface normal. This simplifies to

$$I = \int_{P} J_s \tag{B.53}$$

using the 1-form  $J_s$ . Note that  $J_s$  changes sign depending on the labeling of regions one and two; this ambiguity is equivalent to the existence of two choices for  $\hat{\mathbf{n}}$  in Eq. (B.52).

The following example illustrates the boundary condition on the magnetic field intensity.

## Example 6. Surface current on a sinusoidal surface

A sinusoidal boundary given by  $z - \cos y = 0$  has magnetic field intensity  $H_1 = dx$  A above and zero below. The surface normal 1-form is

$$n = \frac{\sin y \, dy + \, dz}{\sqrt{1 + \sin^2 y}}$$

By the boundary conditions given above,

$$J_s = n \, \lrcorner (n \land dx)$$
  
=  $\frac{1}{1 + \sin^2 y} (\sin y \, dy + dz) \, \lrcorner (\sin y \, dy \, dz + dz \, dx)$   
=  $\frac{dx + \sin^2 y \, dx}{1 + \sin^2 y}$   
=  $dx$  A.

The usual surface current density vector  $\mathbf{J}_{\mathbf{s}}$  is  $(\hat{\mathbf{y}} - \sin y\hat{\mathbf{z}})(1 + \sin^2 y)^{-1/2}$ , which clearly is not dual to dx. The direction of the vector is parallel to the lines of  $J_s$  on the boundary.

# **B.7.4** Surface Charge

The flux density boundary conditions can also be interpreted graphically. Figure B.18a shows the 2-form  $D_1 - D_2$ . The exterior product  $n \wedge (D_1 - D_2)$  yields boxes that have

sides parallel to the boundary, as shown in Fig. B.18b. The component of  $D_1 - D_2$  with tubes parallel to the boundary is removed by the exterior product. The interior product with n removes the surfaces parallel to the boundary, leaving tubes perpendicular to the boundary. These tubes intersect the boundary to form boxes of charge (Fig. B.18c). This is the 2-form  $\rho_s = n \, \lrcorner (n \land (D_1 - D_2))$ .



Figure B.18: (a) The 2-form  $D_1 - D_2$ . (b) The 3-form  $n \wedge (D_1 - D_2)$ , with sides perpendicular to the boundary. (c) The 2-form  $\rho_s$ , represented by boxes on the boundary.

The flux density boundary conditions have as clear a graphical interpretation as those for field intensity: tubes of the difference  $D_1 - D_2$  in electric flux densities on either side of a boundary intersect the boundary to form boxes of surface charge density. Tubes of the discontinuity in magnetic flux density cannot intersect the boundary.

The sign of the charge on the boundary can be obtained from the direction of  $D_1 - D_2$  in region 1 above the boundary, which must point away from positive charge and towards negative charge. The integral of  $\rho_s$  over a surface,

$$Q = \int_{S} \rho_s \tag{B.54}$$

yields the total charge on the surface. Note that  $\rho_s$  changes sign depending on the labeling of regions one and two. This ambiguity is equivalent to the existence of two choices for the area element dA and orientation of the area A in the integral  $\int_A q_s dA$ , where  $q_s$  is the usual scalar surface charge density. Often, the sign of the value of the integral is known beforehand, and the subtlety goes unnoticed.

## **B.8** Conclusion

The primary pedagogical advantages of differential forms are the distinct representations of field intensity and flux density, intuitive graphical representations of each of Maxwell's laws, and a simple picture of electromagnetic boundary conditions. Differential forms provide visual models that can help students remember and apply the principles of electromagnetics. Computational simplifications also result from the use of forms: derivatives are easier to employ in curvilinear coordinates, integration becomes more straightforward, and families of related vector identities are replaced by algebraic rules. These advantages over traditional methods make the calculus of differential forms ideal as a language for teaching electromagnetic field theory.

The reader will note that we have omitted important aspects of forms. In particular, we have not discussed forms as linear operators on vectors, or covectors, focusing instead on the integral point of view. Other aspects of electromagnetics, including vector potentials, Green functions, and wave propagation also benefit from the use of differential forms.

Ideally, the electromagnetics curriculum set forth in this appendix would be taught in conjunction with calculus courses employing differential forms. A unified curriculum, although desirable, is not necessary in order for students to profit from the use of differential forms. We have found that because of the simple correspondence between vectors and forms, the transition from vector analysis to differential forms is generally quite easy for students to make. Familiarity with vector analysis also helps students to recognize and appreciate the advantages of the calculus of differential forms over other methods. We hope that this attempt at making differential forms accessible at the undergraduate level helps to fulfill the vision expressed by Deschamps [33] and others, that students obtain the power, insight, and clarity that differential forms offer to electromagnetic field theory and its applications.