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Power Series Expansions of the Dirichlet Function

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Abstract

This brief report considers the derivation of power series expansions for the so-called Dirichlet function $D_{M,N}(\tau)$ defined as

$$\begin{aligned} D_{M,N}(\tau) &= \frac{\sin(\pi\tau(2M+1)/N)}{\sin(\pi\tau/N)} \\ &= \frac{\sin\alpha\tau}{\sin\beta\tau} \\ &= \frac{\sin ax}{\sin x}, \end{aligned}$$

where N and M are integers with $M < N/2$, $\alpha = \pi(2M+1)/N$, $\beta = \pi/N$, $a = 2M+1$, and $x = \pi\tau/N$. $D_{M,N}(\tau)$ is the bandlimited discrete version of the classic *sine cardinal* or *sinc* function. The purpose of this report is to provide the derivation of the Taylor series for the Dirichlet function. This is compared with a power series derived from a Fourier representation of the Dirichlet function. The later is found to be easier to compute and is more accurate for a given number of terms than the Taylor series.

1 Introduction

The Dirichlet function arises in discrete signal processing as the bandlimited version of the sinc function and is fundamental in bandlimited signal reconstruction [1]. Computing the Dirichlet function for a given argument requires the evaluation of two transcendental sine functions. As an alternate, a Taylor series expansion can be used to evaluate the function at a particular value. The purpose of this report is to provide the derivation of the Taylor series for a general Dirichlet function and compare it to a power series derived from a Fourier representation of the Dirichlet function.

Used in periodic signal processing, the Dirichlet kernel plays an analogous role with the sinc function in continuous signal processing. A discrete, bounded, band-limited periodic signal $g[n]$ can be written as [1]

$$g[n] = \sum_{k=0}^{2M} g_k D_{M,N}(n-kd) \qquad (1)$$

where[1] $g_k = g(kdT)$ is here termed the 'equivalent uniform samples' of the continuous time signal $g(t)$, $d = N/(2M+1)$ is the oversampling factor, T is the sample spacing, and $D_{M,N}(\cdot)$ is the Dirichlet kernel formally defined as

$$D_{M,N}(\tau) \quad = \quad \frac{\sin(\pi\tau(2M+1)/N)}{\sin(\pi\tau/N)}, \tag{2}$$

which is N-periodic in τ and M-band-limited ($M < N/2$). When d is an integer, $g_k = g[kd]$, which can be computed modulo N.

Note that by L'Hospital's rule, when $\sin(\pi\tau/N) = 0$, $D_{M,N}(\tau)$ evaluates to $2M+1$. Thus, we can also write

$$D_{M,N}(\tau) \quad = \quad \begin{cases} \dfrac{\sin(\pi\tau(2M+1)/N)}{\sin(\pi\tau/N)}, & \sin(\pi\tau/N) \neq 0 \\ 2M+1, & \sin(\pi\tau/N) = 0 \end{cases} . \tag{3}$$

The continuous band-limited, periodic signal $g(t)$ can be computed from its equivalent uniform samples using

$$g(t) \quad = \quad \sum_{k=0}^{2M} g_k D_{M,N}(t/T - kd). \tag{4}$$

An illustrative plot of $D_{M,N}(\tau)$ for a particular M and N is shown in Fig. 1. The sinc function-like behavior is apparent. However, unlike a sinc function that rolls off to zero for large arguments, the Dirichlet kernel is periodic with period N. Over one period, e.g., $\tau \in [0 \ldots N]$ or $\tau \in [-N/2 \ldots N/2]$, the sine function in numerator is within the range $[-\pi(2M+1)/2 \ldots \pi(2M+1)/2]$, while the argument of the sine function in the denominator is within the range $[-\pi/2 \ldots \pi/2]$.

The Dirichlet kernel is a special case of the periodic sinc function $D(x; a)$ defined as

$$D(x; a) \quad \triangleq \quad \frac{\sin ax}{\sin x}. \tag{5}$$

Note that $D_{M,N}(\tau) = D(\tau\beta; \alpha)$ where $\beta = \pi/N$ and $\alpha = 2M+1$. Figure 2 illustrates $D(x; a)$ for several values of a.

[1] Following common practice, square brackets are used to denote vaules of a discrete signal while parenthesis are used for continuous time values

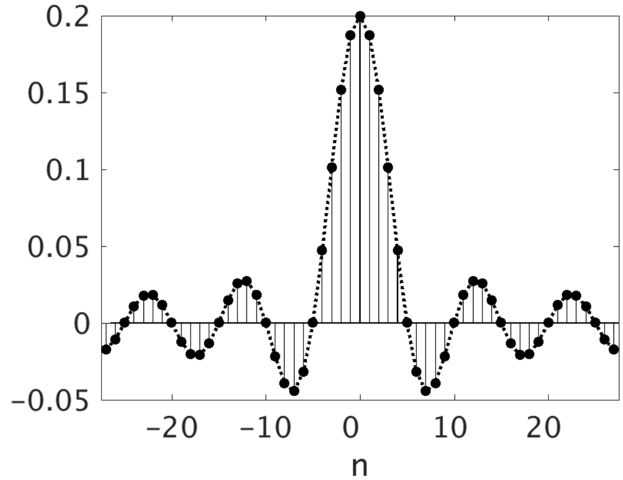

Figure 1: Plot of a ($N = 55$, $M = 5$) Dirichlet kernel evaluated at discrete points. One period is shown.

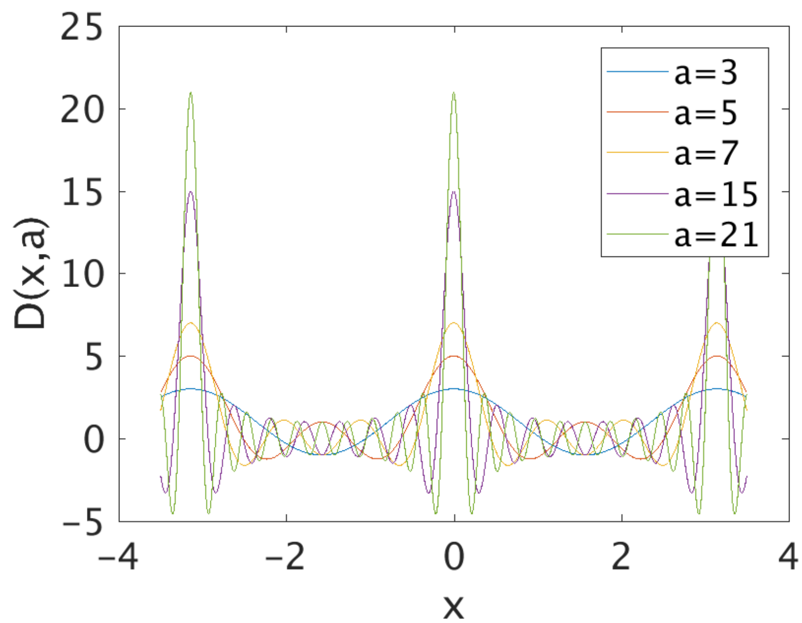

Figure 2: Plot of $D(x; a)$ for several values of a. Note that $D(x; a)$ is periodic in x with period π. Slightly over two periods are shown.

2 Taylor Series

The Taylor Series expansion for a function $f(\tau)$ evaluated at τ relative to the reference τ_0 is

$$\begin{aligned} f(\tau) &= f(\tau_0) + (\tau-\tau_0)f'(\tau_0) + \frac{(\tau-\tau_0)^2}{2!}f''(\tau_0) + \frac{(\tau-\tau_0)^3}{3!}f'''(\tau_0) + \\ &\quad \cdots + \frac{(\tau-\tau_0)^n}{n!}f^{(n)}(\tau_0) + \cdots \qquad (6) \\ &= f(\tau_0) + \sum_{n=1}^{\infty}\frac{(\tau-\tau_0)^n}{n!}f^{(n)}(\tau_0). \qquad (7) \end{aligned}$$

While any appropriate value of τ_0 can be used, we will be interested in the case when $\tau_0 = 0$. This special case of the Taylor series is known as the Maclaurin series.

2.1 Derivatives of the Dirichlet Function

To compute the Taylor series of $D_{M,N}(\tau)$, its derivatives are required. For the case of the Dirichlet function, computing the derivative is complicated by the need for repeated application of L'Hospital's rule, particularly when evaluating the derivatives at zero.

To begin with, let us simplify the notation by using a substitution of variables. Let

$$\begin{aligned} f(x) &\triangleq D_{M,N}(x/\beta) = \frac{\sin ax}{\sin x} \qquad (8) \\ &= \begin{cases} \frac{\sin ax}{\sin x}, & \sin x \neq 0 \\ a, & \sin x = 0 \end{cases}, \qquad (9) \end{aligned}$$

where

$$\begin{aligned} a &\triangleq 2M+1 \qquad (10) \\ \beta &\triangleq \pi/N \qquad (11) \\ \tau &= x/\beta. \qquad (12) \end{aligned}$$

Note that one period of the Dirichlet function corresponds to $x \in [-\pi/2 \ldots \pi/2]$ and that a is a potentially very large integer.

Using the quotient derivative formula,

$$\frac{d}{dx}\frac{u}{v} = \frac{v\frac{d}{dx}u - u\frac{d}{dx}v}{v^2} = \frac{1}{v}\frac{d}{dx}u - \frac{u}{v^2}\frac{d}{dx}v \qquad (13)$$

and noting that

$$\frac{d}{dx}\sin \alpha x = \alpha\cos\alpha x \qquad (14)$$

$$\frac{d}{dx}\sin \beta x = \beta\cos\beta x \qquad (15)$$

and

$$\sin\alpha\cos\beta = \frac{1}{2}\left[\sin(\alpha+\beta)+\sin(\alpha-\beta)\right], \tag{16}$$

the derivative $f'(x)$ of $f(x)$ is

$$\begin{aligned}
f'(x) &= \frac{d}{dx}f(x) && (17)\\
&= \frac{a\sin x\cos ax - \sin ax\cos x}{\sin^2 x} && (18)\\
&= \frac{a\left\{\sin[(a+1)x]+\sin[(1-a)x]\right\} - \left\{\sin[(a+1)x]+\sin[(a-1)x]\right\}}{2\sin^2 x} && (19)\\
&= \frac{(a-1)\sin[(a+1)x)] - (a+1)\sin[(a-1)x]}{2\sin^2 x}. && (20)
\end{aligned}$$

When the value of $f'(x=0)$ is desired, since the numerator and denominator of $f'(x)$ are both zero at $x=0$, L'Hospital's rule must be invoked (twice):

$$\begin{aligned}
f'(0) &= \frac{\frac{d^2}{dx^2}\left\{(a-1)\sin[(a+1)x] - (a+1)\sin[(a-1)x]\right\}\Big|_{x=0}}{\frac{d^2}{dx^2}\left[2\sin^2 x\right]\Big|_{x=0}} && (21)\\
&= \frac{\frac{d}{dx}\left[(a-1)(a+1)\left\{\cos[(a+1)x] - \cos[(a-1)x]\right\}\right]\Big|_{x=0}}{\frac{d}{dx}\left[4\sin x\cos x\right]\Big|_{x=0}} && (22)\\
&= -\frac{\left[(a-1)(a+1)\left\{(a+1)\sin[(a+1)x] - (a-1)\sin[(a-1)x]\right\}\right]\big|_{x=0}}{\left[4\left\{\cos^2(x) - \sin^2(x)\right\}\right]\big|_{x=0}} && (23)\\
&= \frac{0}{4} = 0. && (24)
\end{aligned}$$

Keeping track of the all the terms quickly becomes tedious as the order of the derivative increases. To ameliorate this, we turn to symbolic logic computer programs such as Maple. We write a script to symbolically compute the derivatives and evaluate them at $x=0$. These are then plugged into Eq. 7.

It turns out that all the odd derivatives evaluated at zero are zero-valued[2], so only even-order derivatives are required. We can thus write

$$f(x) = f(0) + \sum_{k=1}^{\infty}\frac{x^{2k}}{(2k)!}f_{2k} \tag{25}$$

where $f_n = f^{(n)}(0)$, or, since $f(0)=a$,

$$f(x) = a + \sum_{k=1}^{\infty}\frac{x^{2k}}{(2k)!}f_{2k}. \tag{26}$$

The f_n are functions of the value of a where $f_n = 0$ for n odd, and $f_2 = \frac{1}{3}(a-a^3)$, $f_4 = \frac{1}{5}a^5 - \frac{2}{3}a^3 + \frac{7}{15}a$, $f_6 = -\frac{1}{7}a^7 + a^5 - \frac{7}{3}a^3 + \frac{31}{21}a$, etc. (see Fig. 3). The f_n are sums of odd power of a up to $n+1$. While an analytic formula can probably be derived, in this report numerical techniques are used to compute the f_n.

[2]This is not surprising since $f(x)$ is symmetric about $x=0$.

$$> \textbf{for } n \textbf{ from } 2 \textbf{ by } 2 \textbf{ to } 20 \textbf{ do } eval\left(\frac{diff\left(numer\left(diff\left(\frac{\sin(a x)}{\sin(x)}, x\$n\right)\right), x\$(n+1)\right)}{diff\left(denom\left(diff\left(\frac{\sin(a x)}{\sin(x)}, x\$n\right)\right), x\$(n+1)\right)}, x = 0\right) \textbf{ end do}$$

$$-\frac{1}{3}a^3 + \frac{1}{3}a$$

$$\frac{7}{15}a + \frac{1}{5}a^5 - \frac{2}{3}a^3$$

$$\frac{31}{21}a - \frac{1}{7}a^7 + a^5 - \frac{7}{3}a^3$$

$$\frac{127}{15}a + \frac{1}{9}a^9 - \frac{4}{3}a^7 + \frac{98}{15}a^5 - \frac{124}{9}a^3$$

$$\frac{2555}{33}a - \frac{1}{11}a^{11} + \frac{5}{3}a^9 - 14a^7 + 62a^5 - 127a^3$$

$$\frac{1414477}{1365}a - 2a^{11} + \frac{77}{3}a^9 - \frac{1364}{7}a^7 + \frac{4191}{5}a^5 - \frac{5110}{3}a^3 + \frac{1}{13}a^{13}$$

$$\frac{57337}{3}a - \frac{637}{15}a^{11} + \frac{4433}{9}a^9 - \frac{18161}{5}a^7 + \frac{46501}{3}a^5 - \frac{1414477}{45}a^3 - \frac{1}{15}a^{15} + \frac{7}{3}a^{13}$$

$$\frac{118518239}{255}a - \frac{3224}{3}a^{11} + \frac{36322}{3}a^9 - \frac{265720}{3}a^7 + \frac{5657908}{15}a^5 - \frac{2293480}{3}a^3 + \frac{1}{17}a^{17} - \frac{8}{3}a^{15} + \frac{196}{3}a^{13}$$

$$\frac{5749691557}{399}a - \frac{168402}{5}a^{11} + \frac{1129310}{3}a^9 - \frac{96184436}{35}a^7 + 11696748a^5 - \frac{118518239}{5}a^3 - \frac{1}{19}a^{19} + 3a^{17} - \frac{476}{5}a^{15} + 2108a^{13}$$

$$\frac{91546277357}{165}a - \frac{42913780}{33}a^{11} + \frac{913752142}{63}a^9 - 105827720a^7 + \frac{2251846541}{5}a^5 - \frac{57496915570}{63}a^3 + \frac{1}{21}a^{21} - \frac{10}{3}a^{19} + 133a^{17} - \frac{80104}{21}a^{15} + 82042a^{13}$$

Figure 3: Maple script and output to compute the first 20 even terms of f_n. In order to ensure a simple output form (as determined empirically), the Maple script separately computes the numerator and denominator terms before evaluating them at $x = 0$.

2.2 Numerically Evaluating the Taylor Series Terms for the Dirichlet Function

The symbolic algebra program Maple is used to compute f_n. The Maple input and script output is shown in Fig. 3. The values of $f_n/n!$ are computed for even n. Figure 4 plots the normalized f_n as a function of n for particular values of N and M. Note that the coefficients become very large but then decline as a function of n, and oscillate in sign. In computing the Taylor series (see Eq. 26) each of the f_n terms get multiplied by $x^{2n}/(2k)!$ where $|x| \leq \pi/2$ for one period. Nonetheless, it can be shown that this Taylor series converges because $|f_n|/(n!)$ rapidly falls off to zero.

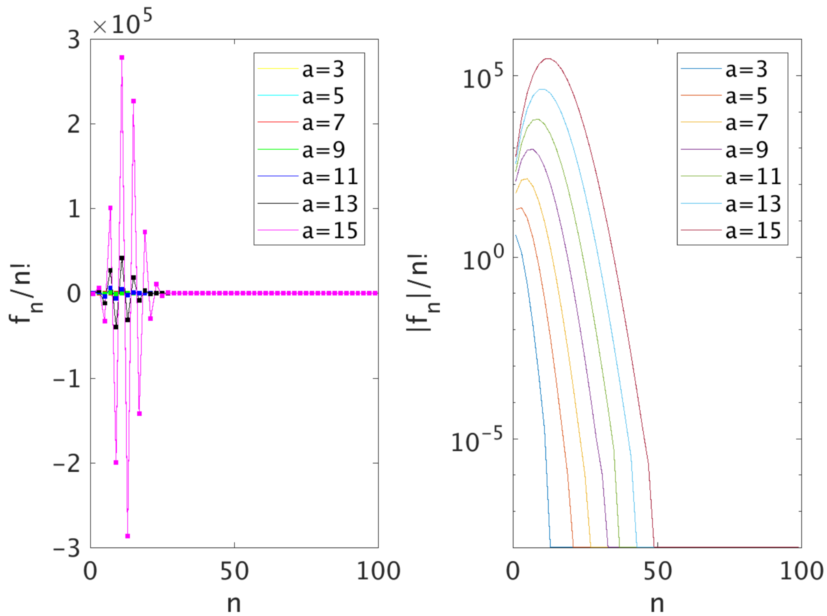

Figure 4: (left) Plot of $f_n/n!$ versus n for several values of $a = 2M + 1$. Note the alternating signs of the terms. (right) Plot of $|f_n|/n!$ versus n for several values of $a = 2M + 1$. Generally, the value initially grows with n, then rapidly falls off.

3 Fourier Transform-Based Series

As an alternate approach to deriving a power series expansion, note that the Dirichlet kernel can be written as [1]

$$D_{M,N}(\tau) = \sum_{k=-M}^{M} W_N^{-k\tau} \quad (27)$$

$$= \sum_{k=-M}^{M} e^{j2\pi k\tau/N}, \quad (28)$$

where W_N is defined as

$$W_N = e^{-j2\pi/N}. \quad (29)$$

This is a consequence of the fact that $D_{M,N}(\tau)$ is the inverse discrete Fourier Transform of a periodic "square wave" of period N and pulse length $2M + 1$. Figure 5 illustrates a periodic discrete square wave and the corresponding Dirichlet kernel.

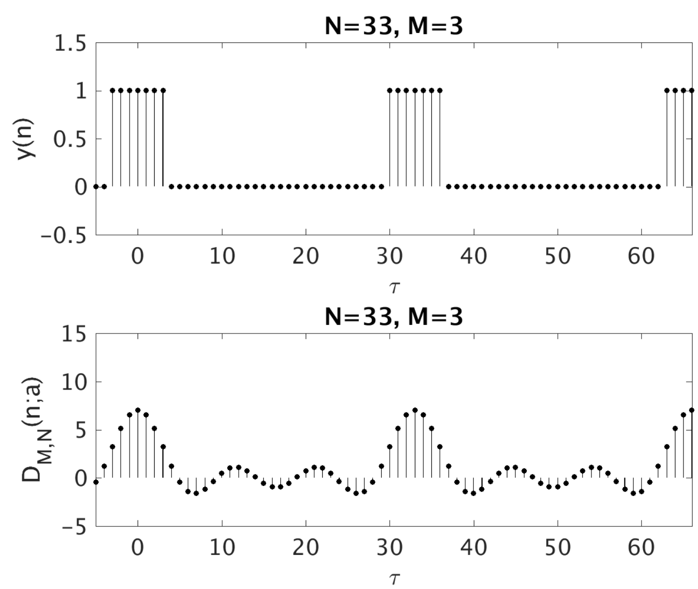

Figure 5: (top) Plot of approximately two periods of a periodic square wave with period $N = 33$ and pulse width $M = 3$. (bottom) Approximately two periods of corresponding Dirichlet kernel.

Using Euler's formula $e^{jx} = \cos x + j \sin x$, $\sin x = -\sin x$, and $\cos x = \cos(-x)$, Eq. 28 becomes

$$D_{M,N}(\tau) = \sum_{k=-M}^{M} [\cos(2\pi k\tau/N) + j\sin(2\pi k\tau/N)] \quad (30)$$

$$= \quad 1 + 2\sum_{k=1}^{M} \cos(2\pi k\tau/N). \tag{31}$$

Substituting the power series for $\cos x$,

$$\cos x \quad = \quad \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}, \tag{32}$$

into Eq. 31 and simplifying,

$$D_{M,N}(\tau) \quad = \quad 1 + 2\sum_{k=1}^{M}\sum_{n=0}^{\infty} (-1)^n \frac{(2\pi k\tau/N)^{2n}}{(2n)!} \tag{33}$$

$$= \quad 1 + 2\sum_{n=0}^{\infty}\left(\sum_{k=1}^{M} k^{2n}\right)(-1)^n \frac{(2\pi\tau/N)^{2n}}{(2n)!} \tag{34}$$

$$= \quad 1 + 2\sum_{n=0}^{\infty} (-1)^n \frac{r_n}{(2n)!}\left(\frac{2\pi}{N}\right)^{2n} \tau^{2n} \tag{35}$$

$$= \quad 1 + 2\sum_{n=0}^{\infty} t_n \tau^{2n}, \tag{36}$$

where

$$r_n \quad = \quad \sum_{k=0}^{M} k^{2n} \tag{37}$$

$$t_n \quad = \quad (-1)^n \frac{r_n}{(2n)!}\left(\frac{2\pi}{N}\right)^{2n}. \tag{38}$$

With these results, and recalling Eqs. 8 and 11, it follows that

$$\frac{\sin ax}{\sin x} = 1 + 2\sum_{n=0}^{\infty} (-1)^n \frac{r_n 4^n}{(2n)!} x^{2n}. \tag{39}$$

A plot of t_n versus n is shown in Fig. 6. Note that compared to the Taylor series the coefficients are much smaller and with their rapid convergence to zero, fewer coefficients are required to obtain the same accuracy in the power series. Hence, this series approach is recommended over the Taylor series.

Equation 37 is an example of a *power sum* and has the analytic solution [2]

$$r_n \quad = \quad \sum_{k=0}^{M} k^{2n} \tag{40}$$

$$= \quad \zeta(-2n) - \zeta(-2n; 1 + M) \tag{41}$$

$$= \quad H_n^{(-2n)} \tag{42}$$

where $\zeta(z)$ is the Riemann zeta function, $\zeta(z; a)$ is the Hurwitz zeta function and $H_n^{(k)}$ is the generalized harmonic number. Other possibilities for expressing r_n exist [2]. Plots of r_n for various M and n are shown in Fig. 7.

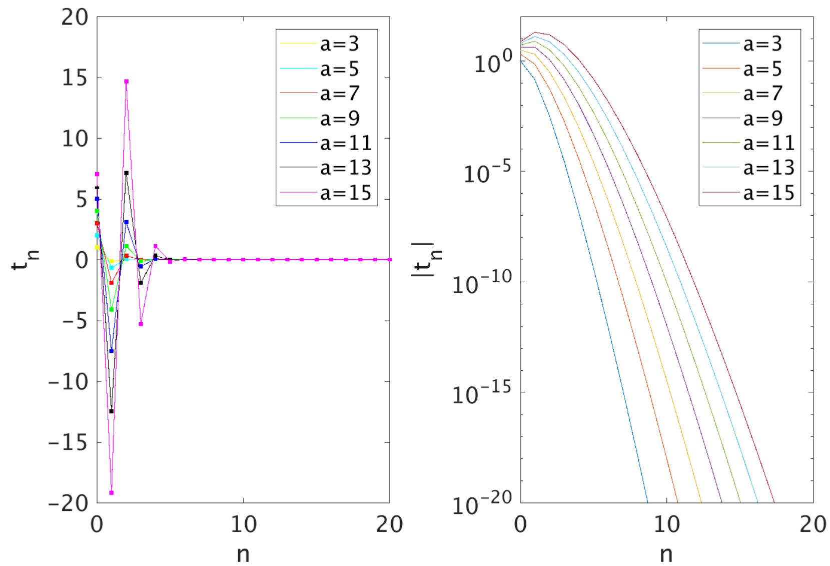

Figure 6: (left) Plot of t_n versus n for several values of $a = 2M + 1$. Note the alternating signs of the terms. (right) Plot of $|t_n|$ versus n for several values of a. Generally, the value initially grows with n, then rapidly falls off. Compare Fig. 4. Note the smaller coefficients and more rapid fall-off of the summed cosine series compared to the Taylor series.

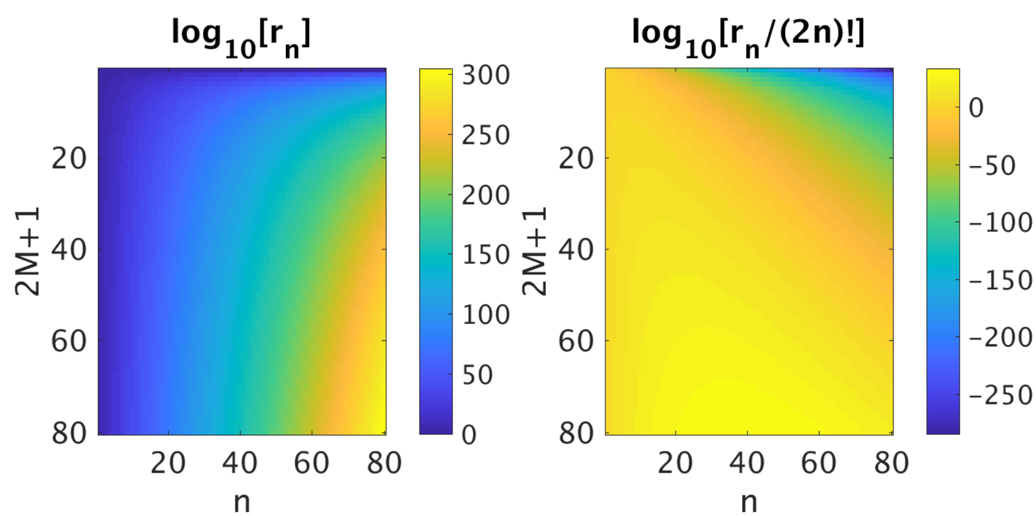

Figure 7: (left) Plot of $\log_{10} r_n$ versus n and $a = 2M+1$. (right) Plot of $\log_{10}[r_n/(2n)!]$ versus n and $a = 2M + 1$.

4 Summary

This report has considered the derivation of two power series for the Dirichlet kernal. The classic Taylor series is difficult to present analytically, but is summarized as

$$D_{M,N}(x/\beta) = \frac{\sin ax}{\sin x} \tag{43}$$

$$= a + \sum_{k=1}^{\infty} \frac{x^{2k}}{(2k)!} f_{2k}, \tag{44}$$

where $a = 2M + 1$, $\beta = \pi/N$, and the first few f_n terms are shown in Fig. 3.

An alternate summed-cosine approach that is based on a Fourier series representation has also been considered. It is easier to express analytically, and is more accurate for a given number of terms than the Taylor series. It is summarized as

$$D_{M,N}(x/\beta) = \frac{\sin ax}{\sin x} \tag{45}$$

$$= 1 + 2\sum_{n=0}^{\infty} (-1)^n \frac{r_n 4^n}{(2n)!} x^{2n}, \tag{46}$$

where

$$r_n = \sum_{k=0}^{M} k^{2n}. \tag{47}$$

References

[1] D.G. Long and R.O.W. Franz, "Band-Limited Signal Reconstruction from Irregular Samples with Variable Apertures," *IEEE Transactions on Geoscience and Remote Sensing*, Vol. 54, No. 4, pp. 2424-2436, doi:10.1109/TGRS.2015.2501366, 2016

[2] `mathworld.wolfram.com/PowerSum.html`, accessed 16 Jul 2019.