CORRESPONDENCES BETWEEN ELECTROMAGNETIC WAVE THEORY AND SHALLOW WATER THEORY

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Abstract - **Although electromagnetic waves are very different from waves in fluids, correlations can be drawn between the two. The common fluid waves considered are longitudinal and transverse waves, characterized by motion either parallel or perpendicular to the direction of propagation. Acoustic waves are longitudinal waves which have limited similarities to transverse electromagnetic waves, while shallow water waves are characterized by transverse motion of the fluid. Strong analogies can be drawn between shallow water waves and electromagnetic waves. Equations for electromagnetic waves and shallow water waves are derived in this paper and a common mathematical form is found to comprise both types of waves. The general form is then examined for several cases as the parameters of the equation are varied. Such examination reveals the similarity between the theories.**

I. INTRODUCTION

Electromagnetic (EM) waves have frequently been discussed by analogy with waves in fluids. The major difficulty with this analogy is that while EM waves are transverse (that is, the fields are perpendicular to the direction of wave propagation), acoustic waves are longitudinal (the analogy between acoustic waves and electromagnetic waves is explored in (Staelin, et al, 1994)). On the other hand, examination of shallow water (SW) wave theory reveals a strong analogy with EM waves since they are transverse. SW theory is an approximation of wave behavior with severe limitations placed on the transmission media, specifically that the fluid depth is very small compared to horizontal deviations (Pedlosky, 1987). Such an assumption implies that the fluid density is constant with time; this forbids acoustic waves.

To describe a wave in SW theory, we first present the geometry of the system. In Fig. (1) a layer of fluid has been plotted in a coordinate system where the horizontal plane is the x-y plane and the vertical direction corresponds to z. The bottom boundary of the fluid is a solid floor described by the function h_B, indicating some height above a constant absolute bottom. The surface of the fluid is a distance H above the bottom. We can consider the surface when no waves are present, so that the surface is some nominal distance H_0 above the bottom; then waves can be described as riding on the nominal surface. By considering the shallow water case, we have restricted our universe to two dimensions; that is, we require that the nominal depth is much shorter than a wavelength, specifically, $H_0 < 0.07\lambda$ (Kundu, 1990).

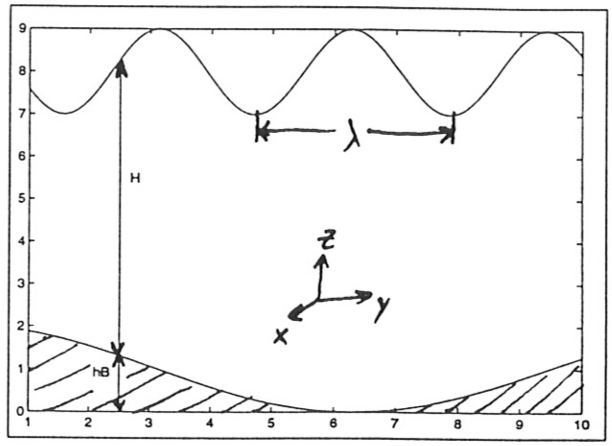

Figure 1. Geometry for shallow water waves

A wave in EM theory is described in terms of either **E** or **H**, the electric or magnetic field intensity, which is a vector field in three dimensions. To restrict this to two dimensions, consider an EM wave linearly polarized so that **H** has only a z component, travelling in the x-y plane. In this two-dimensional universe, the three-dimensional field is simply a scalar function describing the intensity at a point (x,y).

In the next section, wave equations will be derived for shallow water theory and for electromagnetic theory. Section III will examine these equations for a few special cases to reveal some correspondences between the theories.

II. DERIVATION OF THE WAVE EQUATIONS

Maxwell's laws govern EM theory, and from them an equation describing the motion of a wave can be derived. Similarly, fundamental conservation laws govern fluid dynamics, and these yield a wave equation for waves in fluids. To yield the shallow water approximation, we restrict the fluid dynamics wave equation to low-amplitude waves to yield linear behavior, require that the density of the fluid is constant in time, and that the fluid depth is small compared to a wavelength. In this section, we derive wave equations, using the assumptions stated. From the equations, we see that a single equation can be written as a generalization for both.

A. *EM Wave Equations*

The governing equations of EM theory in a source-free region are Maxwell's laws:

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \qquad (1)$$

$$\nabla \times \mathbf{H} = \frac{\partial \mathbf{D}}{\partial t} \quad (2)$$

$$\nabla \cdot \mathbf{D} = 0 \quad (3)$$

$$\nabla \cdot \mathbf{B} = 0. \quad (4)$$

To find a wave equation, we must also specify some constitutive relations for the field quantities; for linear media:

$$\mathbf{D} = \bar{\bar{\epsilon}} \cdot \mathbf{E} \quad (5)$$

$$\mathbf{B} = \bar{\bar{\mu}} \cdot \mathbf{H} \quad (6)$$

The parameters, $\bar{\bar{\epsilon}}$ and $\bar{\bar{\mu}}$ are, in general, complex tensors which depend on frequency and are determined by the properties of the material through which a wave is progressing. To simplify the discussion, let $\bar{\bar{\mu}}$ be a constant scalar.

For the simple case of $\bar{\bar{\epsilon}}$ equal to a constant, a wave equation can be derived for the magnetic field, assuming time-harmonic waves and two-dimensional flow (so that $\frac{\partial}{\partial t} = -i\omega$ and $\frac{\partial}{\partial z} = z$):

$$\frac{\partial^2 H}{\partial x^2} + \frac{\partial^2 H}{\partial y^2} + \frac{\omega^2}{c^2} H = 0, \quad (7)$$

where $c = (\mu\epsilon)^{-1/2}$ is the speed of light squared.

An electron plasma is dispersive, with permitivity depending on the frequency

$$\epsilon(\omega) = \epsilon_0 \left(1 - \frac{\omega_p^2}{\omega^2}\right). \quad (8)$$

With this form for the permitivity, the wave equation becomes

$$\frac{\partial^2 H}{\partial x^2} + \frac{\partial^2 H}{\partial y^2} + \left(\frac{\omega^2 - \omega_p^2}{c^2}\right) H = 0. \quad (9)$$

Let's further complicate the plasma by applying an external magnetic field, $\vec{B}_0$ in the z direction (a gyrotropic medium). This requires the permitivity to be expressed as a complex tensor:

$$\bar{\bar{\epsilon}} = \begin{pmatrix} \epsilon & i\epsilon_g & 0 \\ -i\epsilon_g & \epsilon & 0 \\ 0 & 0 & \epsilon_z \end{pmatrix}$$

where

$$\epsilon = \epsilon_0 \left[1 - \frac{\omega_p^2}{(\omega^2 - \omega_c^2)}\right] \quad (10)$$

$$\epsilon_g = \epsilon_0 \left[\frac{-\omega_p^2 \omega_c}{\omega(\omega^2 - \omega_c^2)}\right] \quad (11)$$

$$\epsilon_z = \epsilon_0 \left(1 - \frac{\omega_p^2}{\omega^2}\right) \quad (12)$$

$$\omega_c = \frac{qB_0}{m}. \quad (13)$$

A complete description of the wave equation would obviously be complicated, but it would certainly result in a complex equation. The solution, based on a magnetic field in the z direction and propagation in the x-y plane, is two waves travelling in the same direction but with different dispersion relations (Kong, 1990). Specifically,

$$\omega^2 = \frac{k^2}{2\mu} \left[(\kappa + \kappa_z) \pm (\kappa - \kappa_z)\right], \quad (14)$$

where κ and κ_z are found from the inverse of the permitivity tensor.

B. SW Wave Equations

The fundamental equations of motion for a fluid are based on conservation of mass, momentum and energy (Pedlosky, 1987). The density of the fluid is specified by ρ, while $\mathbf{u}$ describes the flow velocity in three dimensions. The pressure is represented by p, and ϕ describes conservative body forces such as gravity. Ω is a vector describing the rotation of the earth so that the coriolis effect can be included. For the special case that friction is neglected and the density is considered constant with time, $\frac{d\rho}{dt} = 0$, only two equations are needed to specify the equations of state, namely

$$\nabla \cdot \mathbf{u} = 0 \quad (15)$$

$$\frac{d\mathbf{u}}{dt} + 2\Omega \times \mathbf{u} = -\frac{\nabla p}{\rho} + \nabla\phi \quad (16)$$

These equations are still harder than we want to solve. If we make some approximations to neglect some terms then the equations should still be general enough to be applicable to a wide class of problems, but simple enough to be tractable.

Now we assume that the only body force, ϕ, is gravity. The coriolis parameter describes the effect of the earth's rotation and is defined as $f = 2\Omega \sin\theta$, and the total derivative is expanded as $\frac{d\mathbf{u}}{dt} = \frac{\partial \mathbf{u}}{\partial t} + \mathbf{u}\nabla \cdot \mathbf{u}$. The hydrostatic approximation suggests that the pressure at any point is equal to the weight of the column of fluid above that point, plus a constant, p_0

$$p(x, y, z, t) = \rho g(h - z) + p_0. \quad (17)$$

Further, assuming that the vertical height of the fluid is much smaller than the horizontal extent, the horizontal components of the fluid velocity become independent of z. The boundary conditions require that $w = 0$ at the bottom ($z = h_B$) and $w = \frac{dh}{dt}$ at the surface. Defining the total depth as $H = h - h_B$ allows the conservation equations to be rewritten as

$$\frac{\partial H}{\partial t} + \frac{\partial}{\partial x}(uH) + \frac{\partial}{\partial y}(vH) = 0 \quad (18)$$

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} - fv + g\frac{\partial h}{\partial x} = 0 \quad (19)$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + fu + g\frac{\partial h}{\partial y} = 0 \quad (20)$$

Allowing a time-varying disturbance on the surface suggests defining $H(x, y, t) = H_0(x, y) + \eta(x, y, t)$ where η is the height above the nominal value of the fluid depth, H_0, at a point:

$$\frac{\partial \eta}{\partial t} + \frac{\partial}{\partial x}(uH_0 + u\eta) +$$

$$\frac{\partial}{\partial y}(vH_0 + v\eta) = 0 \quad (21)$$

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} - fv + g\frac{\partial \eta}{\partial x} = 0 \quad (22)$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + fu + g\frac{\partial \eta}{\partial y} = 0. \quad (23)$$

These are non-linear equations which can be used (theoretically) to obtain the surface disturbance and horizontal velocities of the fluid. In practice, this is a very difficult problem, but a linear approximation still provides considerable insight. To linearize them, we consider only small amplitude variations so higher order terms can be ignored:

$$\frac{\partial \eta}{\partial t} + \frac{\partial}{\partial x}(uH_0) + \frac{\partial}{\partial y}(vH_0) = 0 \quad (24)$$

$$\frac{\partial u}{\partial t} - fv + g\frac{\partial \eta}{\partial x} = 0 \quad (25)$$

$$\frac{\partial v}{\partial t} + fu + g\frac{\partial \eta}{\partial y} = 0. \quad (26)$$

A differential equation in the single variable η can be obtained by manipulating these three equations:

$$\frac{\partial}{\partial t}\left[\left(\frac{\partial^2}{\partial t^2} + f^2\right)\eta - \nabla\cdot(C_0^2\nabla\eta)\right] - gfJ(H_0,\eta) = 0 \quad (27)$$

where $C_0^2 = gH_0$ and the Jacobian term is defined as:

$$J(A,B) = \frac{\partial A}{\partial x}\frac{\partial B}{\partial y} - \frac{\partial A}{\partial y}\frac{\partial B}{\partial x}. \quad (28)$$

The velocities u and v can be found as solutions to the following differential equations:

$$\left(\frac{\partial^2}{\partial t^2} + f^2\right)u = -g\left(\frac{\partial^2 \eta}{\partial x \partial t} + f\frac{\partial \eta}{\partial y}\right), \quad (29)$$

$$\left(\frac{\partial^2}{\partial t^2} + f^2\right)v = -g\left(\frac{\partial^2 \eta}{\partial y \partial t} + f\frac{\partial \eta}{\partial x}\right). \quad (30)$$

Equation (27) provides the time-varying wave displacement, η, in terms of the coriolis parameter, f, the fundamental speed, C_0, the gravitational constant, g, and the nominal surface height, H_0, all of which are constants at a point (x,y). As with EM, we will assume time-harmonic waves in rectangular coordinates to simplify the equation.

C. *Generalized Wave Equations*

The phenomenon of waves has been observed in many realms of science, and various equations have been used to mathematically model the behavior. The equation for EM and SW waves can be written in a generalized form, assuming time-harmonic waves:

$$\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} + iA_1\frac{\partial \psi}{\partial x} + iA_2\frac{\partial \psi}{\partial y} + A_3\psi = 0, \quad (31)$$

where ψ represents the field of interest, and the constants, A_1, A_2, and A_3, depend on the constitution of the medium.

For shallow water theory, ψ relates to η, the fluid height above some nominal value and the constants are

$$A_1 = -\frac{f}{\omega H_0}\frac{\partial H_0}{\partial y} \quad (32)$$

$$A_2 = \frac{f}{\omega H_0}\frac{\partial H_0}{\partial x} \quad (33)$$

$$A_3 = \left(\frac{\omega^2 - f^2}{C_0^2}\right). \quad (34)$$

For electromagnetic theory, the wave equation is derived from Maxwell's Laws. The constants for electromagnetic theory will depend on the medium, as did the shallow water constants, and solutions to the wave equation will be similar. For example, the constants

$$A_1 = 0 \quad (35)$$

$$A_2 = 0 \quad (36)$$

$$A_3 = \left(\frac{\omega^2 - \omega_p^2}{c^2}\right) \quad (37)$$

apply to isotropic media, including dispersive (plasma-like) media by allowing ω_p to be non-zero. For more complicated media (such as anisotropic), the constants are more complicated.

III. SOLUTIONS TO THE WAVE EQUATION

In this section, we will examine the general equation for waves, Eq. (31), specifically for various SW circumstances, and see how these correspond to EM circumstances.

A. *Plane Waves with $A_1 = A_2 = 0$*

Requiring that $A_1 = A_2 = 0$ suggests for SW that the bottom is flat, while for EM that the medium is isotropic, but possibly dispersive. The generalized wave equation [Eq. (31)] becomes

$$\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} + A_3\psi = 0. \quad (38)$$

Assuming the solution is separable as $\psi(x,y) = X(x)Y(y)$, then

$$\frac{X''}{X} = -\gamma_x^2 \quad (39)$$

$$\frac{Y''}{Y} = -\gamma_y^2, \quad (40)$$

where γ_x and γ_y are complex constants. The solutions to these second-order differential equations in an unbounded medium are plane waves, so the full solution, $\psi(x,y) = \psi_0 e^{-\gamma_x x - \gamma_y y}$, represents a plane wave traveling in the x-y plane.

The constants γ_x and γ_y are the propagation constants in the x and y directions, respectively; they must satisfy the constraint that $\gamma_x^2 + \gamma_y^2 + A_3 = 0$.

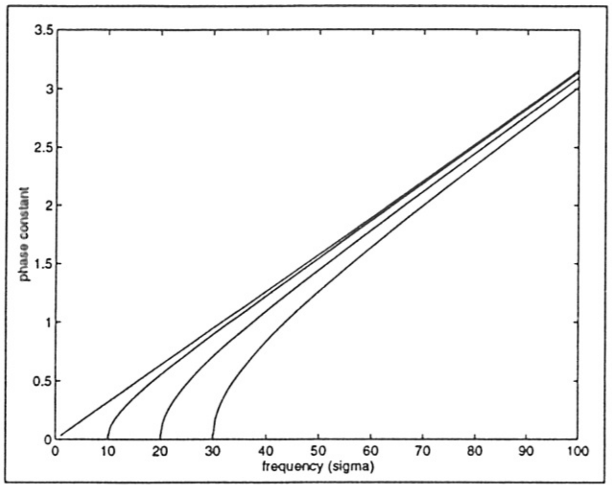

Figure 2. The dispersion diagram when H_0 is a constant (for various values of f)

In general, γ_x and γ_y are complex. If they are real, we have the uninteresting case of attenuation (uninteresting because we are interested in waves that propagate a long distance). But if these constants are imaginary, $\gamma_x = ik_x$ and $\gamma_y = ik_y$, then the wavenumber, $k = \sqrt{k_x^2 + k_y^2} = \sqrt{A_3}$ is real; the phase velocity of the wave is defined as the frequency divided by the wavenumber, $\omega/\sqrt{A_3}$. Using the variables from the electromagnetic case,

$$v_p = \frac{c}{\sqrt{1 - (\omega_p/\omega)^2}}. \qquad (41)$$

The dispersion relation provides the relationship between the frequency, ω, and the wavenumber, k; this is found from A_3,

$$\omega^2 = \omega_p^2 + c^2 k^2. \qquad (42)$$

The phase velocity and dispersion relation clearly correspond to EM wave propagation in a parallel plate wave guide in the TM_0 mode (also called the TEM mode) with dispersion. Figure (2) shows the wavenumber as a function of frequency. The cutoff frequency is not normally seen in the TEM mode, though it does appear in rectangular waveguides where additional boundary conditions constrain the wave. The coriolis parameter or plasma frequency provides an additional constraint manifest in a cutoff frequency.

In SW theory, the dispersion diagram describes Kelvin waves (the mode with no cutoff frequency, corresponding to the dispersionless case) and Poincare modes where the cutoff frequency depends on the coriolis parameter.

B. Plane Waves with $A_1 \neq 0$

Generalizing to allow A_1 to be negative implies for SW that the fluid bottom slopes in the positive y direction, and for EM that the medium is anisotropic. Note that with a simple coordinate rotation, this is the same case as $A_2 \neq 0$. Equation (31) (the general wave equation) becomes

$$\frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} + iA_1\frac{\partial\psi}{\partial x} + A_3\psi = 0. \qquad (43)$$

Again, assuming the non-attenuating solution $\psi(x, y) = \psi_0 e^{-i(k_x x + k_y y)}$, the dispersion relation is

$$k_x^2 + k_y^2 - A_1 k_x - A_3 = 0. \qquad (44)$$

If the wave is travelling up the slope, $k_x = 0$, and we have simply that $k_y^2 = A_3$. That is, this is simply the wave travelling at its characteristic velocity based on the paramters of the fluid. However, if the wave is travelling across the slope, $k_y = 0$, then $k_x = \frac{A_1}{2} \pm \sqrt{\left(\frac{A_1}{2}\right)^2 + A_3}$. This implies that travelling across a slope in the positive x direction allows two possible speeds of propagation. While not a direct analogy with gyrotropic media, this does show similar behavior in that there are two types of waves.

IV. DISCUSSION

The wave equations based on electromagnetic theory and shallow water theory have been examined and, upon examination, the similarities between the theories are striking.

The general form of a wave equation applies quite well to both theories. Because we are looking at travelling waves, we would expect each theory to have a frequency, ω, and a fundamental speed based on the parameters of the medium, $c = (\mu\epsilon)^{-1/2}$ or $C_0 = \sqrt{gH_0}$. But what is rather surprising is the correspondence between dispersive materials. Because SW is subject to the rotation of the earth, an additional constraint based on the coriolis parameter, f, establishes a cutoff frequency below which waves will not propagate. Similarly, plasma-like EM media are dispersive based on the plasma frequency, ω_p. The mathematical forms of these effects are identical.

In EM theory, gyrotropic materials are cumbersome to deal with mathematically and difficult to visualize intuitively. While the analogy is not exact, SW waves which depend on the topography of the bottom provide a similar type of effect in that motion across the slope yields two types of waves; that is, two waves with different speeds.

SW theory provides strong intuition for EM theory because we can see water waves and recognize that the fluid height corresponds to the magnetic field intensity of a similar EM problem. The general wave equation is readily applied to bounded systems such as rectangular waveguides for EM or channels for SW.

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