OSA - SRS 2001

Multidimensional Reconstruction From Irregular Samples

Reinhard O.W. Franz
Brigham Young University
Center for Remote Sensing
Provo, UT 84 602
(801)-378-8173, (801)-378-6586, franz@et.byu.edu
David G. Long
Brigham Young University
Department of Electrical and Computer Engineering
Provo, UT 84 602
(801)-378-4383, (801)-378-6586, long@ee.byu.edu
http://www.cers.byu.edu

Abstract: We extend Gröchenig's solution for the irregular sampling problem for bandlimited functions. We show that the algorithm can be extended to functions on a n-dimensional domain when only local averages of the function are known.

OCIS codes: (100.3010) Image reconstruction techniques

1 Introduction

In 1992, K. Gröchenig published a constructive solution to the so-called "irregular sampling problem" (cf. [1]) which appears to be of utmost importance for the applied sciences, in particular for the signal processing community. The algorithms presented require sampling densities close to the Nyquist rate, require no separation conditions, converge reasonably fast and allow easy estimates for the relative approximation error.

While Gröchenig focuses his attention on the 1-dimensional case and only briefly considers irregular sampling in higher dimensions, we would like to study this case and consider the more general situation of reconstruction of bandlimited functions with n-dimensional domain $\mathbb{R}^n$ from local averages on a "δ-dense" subset X contained in $\mathbb{R}^n$. Such local averages correspond to aperture filtered sampling. Gröchenig's proof can be extended to this situation. These results will be of use in many settings in image reconstruction.

2 The Reconstruction Algorithm

The core of the reconstruction algorithm discussed here is the following lemma concerning the existence and nature of inverse operators presenting the so-called "Neumann expansion" for an invertible operator:

Lemma 2.1 *Suppose $B = (B, \|\cdot\|)$ is a Banach space. Let $\mathcal{B}(B) := (\mathcal{B}(B), \|\cdot\|')$ denote the associated Banach algebra of all bounded linear operators on B, where $\|\cdot\|'$ denotes the operator norm on B. Then if $A \in \mathcal{B}(B)$ is a bounded linear operator on B such that $\|\mathrm{Id} - A\|' < 1$, where Id denotes the identity operator on B, then A is invertible on B and $A^{-1} \in \mathcal{B}(B)$ and A^{-1} has the expansion*

$$A^{-1} = \sum_{n=0}^{\infty} (\mathrm{Id} - A)^n.$$

Corollary 2.1 *With the same assumptions as in the previous lemma, every $f \in B$ can be reconstructed by the following iteration procedure*

$$\phi_0 = A(f)$$
$$\phi_{n+1} = \phi_n - A(\phi_n)$$

and

$$f = \sum_{n=0}^{\infty} \phi_n.$$

Moreover, with $f_n := \sum_{k=0}^{n} \phi_k$, *the approximation error can be estimated by*

$$\|f - f_n\| \leq (\|\operatorname{Id} - A\|')^{n+1} \frac{1 + \|\operatorname{Id} - A\|'}{1 + \|\operatorname{Id} - A\|'} \|f\|.$$

We will now apply these techniques in the setting of image reconstruction and chose here the continuous model of the square-integrable functions on the n-dimensional space $\mathbb{R}^n$. In a future paper we will study the discrete case.

Let $L^2(\mathbb{R}^n)$ denote the Hilbert space of all square-integrable functions on $\mathbb{R}^n$ with scalar product and norm

$$\langle f, g \rangle := \int_{\mathbb{R}^n} f(x) g(x)\, dx, \quad \|f\| := \left(\int_{\mathbb{R}^n} |f(x)|^2\, dx \right)^{\frac{1}{2}},$$

respectively. Moreover, let $\Omega \subseteq \mathbb{R}^n$ be a compact set and $\omega := (\omega_1, \ldots, \omega_n)$ its extension with $\omega_i := \sup_{t \in \Omega} |t_i|$, where s_i denotes the i-th coordinates of $s \in \mathbb{R}^n$ as usual. Let

$$B^2(\Omega) := \{ f \in L^2(\mathbb{R}^n) \mid \operatorname{supp}(\hat{f}) \subseteq \Omega \}$$

denote the closed subspace of all bandlimited functions in $L^2(\mathbb{R}^n)$ whose spectrum is contained in Ω, where the Fourier transform $\hat{f}$ is defined by

$$\hat{f}(\xi) := \int_{\mathbb{R}^n} f(x) e^{-j(x \cdot \xi)}\, dx.$$

Note that $B^2(\Omega)$ as a closed subspace of $L^2(\mathbb{R}^n)$ is also a Hilbert space, and in particular, a Banach space. Finally, let P denote the orthogonal projection from $L^2(\mathbb{R}^n)$ onto $B^2(\Omega)$, defined by $\widehat{P(f)} = \chi_\Omega \cdot \hat{f}$, where χ_Ω is the characteristic function of the set Ω. The following lemma is due to S. Bernstein:

Lemma 2.2 *If* $f \in L_2(\mathbb{R}^n)$ *is bandlimited and* $\operatorname{supp}(\hat{f}) \subseteq \prod_{i=1}^{n} [-\omega_i, \omega_i]$ *then* f *is an entire function and*

$$\|D^\alpha f\| \leq \omega^\alpha \|f\|$$

for all $\alpha = (\alpha_1, \ldots, \alpha_n) \in (\mathbb{N}_0)^n$, *where* $\mathbb{N}_0 := \{0, 1, 2, \ldots\}$ *denotes the set of all nonegative integers.*

Suppose $\delta := (\delta_1, \ldots, \delta_n) \in \mathbb{R}^n$ with $\delta_i > 0$ for $i = 1, \ldots, n$. Following Gröchenig, we define a family $X = (x_i)_{i \in I}$ of points in $\mathbb{R}^n$ to be *δ-dense* if

$$\bigcup_{i \in I} B_\delta(x_i) = \mathbb{R}^n.$$

$B_\delta(x)$ denotes the cube $\prod_{i=1}^{n} [\xi_i - \delta_i/2, \xi_i + \delta_i/2]$ centered at the point $x = (\xi_1, \ldots, \xi_n) \in \mathbb{R}^n$. Let $(\psi_i)_{i \in I}$ be a family of functions $\psi_i : \mathbb{R}^n \longrightarrow \mathbb{R}$ such that $0 \leq \psi_i(x) \leq 1$, $\sum_{i \in I} \psi_i(x) = 1$ and $\operatorname{supp}(\psi_i) \subseteq B_\delta(x_i)$ for all $i \in I$ and $x \in \mathbb{R}^n$. Finally, let $(u_i)_{i \in I}$ denote an arbitrary family of functions $u_i : \mathbb{R}^n \longrightarrow \mathbb{R}$ satisfying the conditions $\operatorname{supp}(u_i) \subseteq B_\delta(x_i)$, $0 \leq u_i(x) \leq 1$ and $\int_{\mathbb{R}^n} u_i(x)\, dx = 1$ for all $i \in I$ and $x \in \mathbb{R}^n$. Given a function $f \in L^2(\mathbb{R}^n)$ we define

$$\int_{\mathbb{R}^n} f(x) u_i(x)\, dx = \langle u_i, f \rangle$$

to be the *local average* of f at x_i with respect to the average function u_i. Then we have the following

Theorem 2.1 *If $\delta = (\delta_1, \ldots, \delta_n)$, $\delta_i > 0$ is chosen such that $\delta \cdot \omega < \ln 2$ and if $X = (x_i)_{i \in I}$ is a δ-dense family of points x_i in $\mathbb{R}^n$, then every bandlimited function $f \in B^2(\Omega)$ can be reconstructed from its local averages $\langle u_i, f \rangle$ at x_i for $i \in I$ using the following iteration procedure*

$$\phi_0 := P\Big(\sum_{i \in I} \langle u_i, f \rangle \psi_i\Big)$$

$$\phi_{n+1} := \phi_n - P\Big(\sum_{i \in I} \langle u_i, \phi_n \rangle \psi_i\Big),$$

for $n \in \mathbb{N}_0$, and

$$f = \sum_{n=0}^{\infty} \phi_n.$$

Moreover, for $f_n := \sum_{j=0}^{n} \phi_j$, we have the following estimate for the approximation error and the rate of convergence of the reconstruction algorithm:

$$\|f - f_n\| \leq \left(e^{\omega \cdot \delta} - 1\right)^{n+1} \frac{e^{\omega \cdot \delta}}{2 - e^{\omega \cdot \delta}} \|f\|$$

3 Conclusion

Theorem 2.1 suggests that a signal can be completely recovered from sampled averages for δ-dense samples so long as the signal is bandlimited such that $\omega \cdot \delta < \ln 2$ and the aperture function is narrow enough. The result is applicable for both irregular and regular sampling but is of most interest in the irregular sampling case since it suggests that if the sampling is adequate, the averaging function has minimal impact on the signal recovery.

References

1. K. Gröchenig, "Reconstruction Algorithms In Irregular Sampling," *Mathematics of Computation*, Vol. 59, No. 199, pp. 181–194, 1992.
2. H. Feichtinger and K. Gröchenig, "Iterative Reconstruction of Multivariate Band-limited Functions from Irregular Sampling Values," *SIAM Journal of Mathematical Analysis*, vol. 23, no. 1, pp. 244-261, 1992.